On discrete time Beverton-Holt population model with fuzzy environment *

Qianhong Zhang\textsuperscript{1,*}, Fubiao Lin\textsuperscript{1}, Xiaoying Zhong\textsuperscript{2}

\textsuperscript{1}School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang, Guizhou 550025, China
\textsuperscript{2}Library, Guizhou University of Finance and Economics, Guiyang, Guizhou 550025, China

Abstract

In this work, dynamical behaviors of discrete time Beverton-Holt population model with fuzzy parameters are studied. It provides a flexible model to fit population data. For three different fuzzy parameters and fuzzy initial conditions, according to a generalization of division (g-division) of fuzzy number, it can represent dynamical behaviors including boundedness, global asymptotical stability and persistence of positive solution. Finally, two examples are given to demonstrate the effectiveness of the results obtained.

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1 Introduction

Discrete time single-species model is the most appropriate mathematical description of life histories of organism whose reproduction occurs only once a year during a very short season. It is assumed that, in the rest of the year, the population is only subjected to mortality, but not to births. Therefore the between year dynamics is characterized by a first order difference equation $x_{n+1} = f(x_n)$, where $x_n$ denotes population at the $n$th generation. The production function $f$ is usually density-dependent, and the strength of density dependence is determined by several parameters including growth rate, the probability of surviving the reproductive season, the carrying capacity of surrounding environment, and intraspecific cooperation or competition factors. These models are widely used in fisheries and many organisms [1]. Beverton-Holt model [2] is one of classic population model which has been studied

$$x_{n+1} = \frac{\beta x_n}{1 + \delta x_n}, n = 0, 1, \ldots,$$

where $x_n$ is population at the $n$th generation, $\beta$ represents a productivity parameter, and $\delta$ controls the level of density dependence. Since then, many results on the model and the generation of the model have been widely obtained by some researchers [3,4,5].

In fact, the identification of the parameters of the model is usually based on statistical method, starting from data experimentally obtained and on the choice of some method adapted to the identification. These models, even the classic deterministic approach, are subjected to inaccuracies (fuzzy uncertainty) that can be caused by the nature of the state variables, by parameters as coefficients of the model and by initial conditions.

In our real life, we have learned to deal with uncertainty. Scientists also accept the fact that uncertainty is very important study in most applications. Modeling the real life problems in such cases, usually involves vagueness or uncertainty in some of the parameters. The concept of fuzzy set and system

\*Corresponding author. Email: zqianhong68@163.com.cn (Q.Zhang)
was introduced by Zadeh [6] and its development has been growing rapidly to various situation of theory and application including fuzzy differential and fuzzy difference equations. It is well known that fuzzy difference equation is a difference equation whose parameters or the initial values are fuzzy numbers, and its solutions are sequences of fuzzy numbers. It has been used to model a dynamical systems under possibility uncertainty [7]. Due to the applicability of fuzzy difference equation for the analysis of phenomena where imprecision is inherent, this class of difference equation is a very important topic from theoretical point of view and also its applications. Recently there has been an increasing interest in the study of fuzzy difference equations (see [8-23]).

Inspired with the previous, by virtue of the theory of fuzzy difference equation, in this work, we consider the following discrete time Beverton-Holt model with fuzzy uncertainty parameters and initial conditions.

\[ x_{n+1} = \frac{Ax_n}{1+Bx_n}, \quad n = 0, 1, \ldots, \tag{1.1} \]

where \( x_n \) is population at the \( n \)th generation, \( A \) denotes a productivity parameter, \( B \) controls the level of density dependence. Furthermore \( A, 1, B \) and the initial value \( x_0 \) are positive fuzzy numbers.

The main aim of this work is to study the existence of positive solutions of Beverton-Holt population model (1.1). Furthermore, according to a generation of division (g-division) of fuzzy numbers, we derive some conditions so that every positive solution of Beverton-Holt population model (1.1) is bounded and persistent. Finally, under some conditions we prove that Beverton-Holt population model (1.1) has a unique positive equilibrium \( x \) and every positive solution tends to \( x \) as \( n \to \infty \).

\section{Preliminary and definitions}

Firstly, we give the following definitions.

\textbf{Definition 2.1.}[24] \( u : R \to [0, 1] \) is said to be a fuzzy number if it satisfies conditions (i)-(iv) written below:

(i) \( u \) is normal, i.e., there exists an \( x \in R \) such that \( u(x) = 1 \);

(ii) \( u \) is fuzzy convex, i.e., for all \( t \in [0, 1] \) and \( x_1, x_2 \in R \) such that

\[ u(tx_1 + (1-t)x_2) \geq \min\{u(x_1), u(x_2)\}; \]

(iii) \( u \) is upper semicontinuous;

(iv) The support of \( u, \text{supp} u = \bigcup_{x \in (0,1]} [u^x_0 = \{ x : u(x) > 0 \} \) is compact.

For \( \alpha \in (0, 1] \), the \( \alpha \)-cuts of fuzzy number \( u \) is \( \{ u \}_\alpha = \{ x \in R : u(x) \geq \alpha \} \) and for \( \alpha = 0 \), the support of \( u \) is defined as \( \text{supp} u = \{ u \}^0_0 = \{ x \in R : u(x) > 0 \} \).

\textbf{Definition 2.2. Fuzzy Number (Parametric form):}[24] A fuzzy number \( u \) in a parametric form is a pair \((\underline{u}, \overline{u})\) of functions \( \underline{u}(r), \overline{u}(r), 0 \leq r \leq 1 \), which satisfies the following requirements:

(1) \( \underline{u}(r) \) is a bounded monotonic increasing left continuous function,

(2) \( \overline{u}(r) \) is a bounded monotonic decreasing left continuous function,

(3) \( \overline{u}(r) \leq \underline{u}(r), 0 \leq r \leq 1 \).

A crisp (real) number \( x \) is simply represented by \((\underline{u}(r), \overline{u}(r)) = (x, x), 0 \leq r \leq 1 \). The fuzzy number space \( \{ (\underline{u}(r), \overline{u}(r)) \} \) becomes a convex cone \( E^1 \) which could be embedded isomorphically and isometrically into a Banach space [24].

\textbf{Definition 2.3.}[24] The distance between two arbitrary fuzzy numbers \( u \) and \( v \) is defined as follows:

\[ D(u, v) = \sup_{\alpha \in [0, 1]} \max\{|u_{l,\alpha} - v_{l,\alpha}|, |u_{r,\alpha} - v_{r,\alpha}|\}. \tag{2.1} \]

It is clear that \((E^1, D)\) is a complete metric space.

\textbf{Definition 2.4.}[24] Let \( u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r)) \in E^1, 0 \leq r \leq 1 \), and arbitrary \( k \in R \). Then

(i) \( u = v \) iff \( \underline{u}(r) = \underline{v}(r), \overline{u}(r) = \overline{v}(r) \),

(ii) \( u + v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)) \),

\[ 2 \]
having level cuts \( [0, 1] \) defined as \( 0 < M; N > 0 \) and \( k \geq 0 \), then there is \( u \)

(iii) \( u - v = (u(r) - v(r), \pi(r) - \nu(r)) \),

(iv) \( ku = \left\{ \left( k u(r), k \pi(r) \right), k \geq 0 \right\} \),

(v) \( uv = (\min\{u(r)v(r), u(r)\pi(r), \pi(r)\nu(r), \nu(r)\pi(r)\}, \max\{u(r)v(r), u(r)\pi(r), \pi(r)\nu(r), \nu(r)\pi(r)\}) \).

**Definition 2.5. Triangular Fuzzy Number.** [24] A triangular fuzzy number (TFN) denoted by \( A \) is defined as \((a, b, c)\) where the membership function

\[
A(x) = \begin{cases} 
0, & x \leq a; \\
\frac{x - a}{b - a}, & a \leq x \leq b; \\
1, & x = b; \\
\frac{c - x}{c - b}, & b \leq x \leq c; \\
0, & x \geq c.
\end{cases}
\]

The \( \alpha \)-cuts of \( A = (a, b, c) \) are described by \( [A]^\alpha = \{ x \in \mathbb{R} : A(x) \geq \alpha \} = [a + \alpha(b - a), c - \alpha(c - b)] = [A_l, A_r, \alpha] \), \( \alpha \in [0, 1] \), it is clear that the \( [A]^\alpha \) are closed interval. A fuzzy number is positive if \( \supp A \subseteq (0, \infty) \).

The following proposition is fundamental since it characterizes a fuzzy set through the \( \alpha \)-levels.

**Proposition 2.1** [24] If \( \{ A^\alpha : \alpha \in [0, 1] \} \) is a compact, convex and not empty subset family of \( \mathbb{R}^n \) such that

(i) \( \bigcup \{ A^\alpha : \alpha \in [0, 1] \} = A^0 \),

(ii) \( A^0 \supseteq A^1 \), if \( \alpha_1 \leq \alpha_2 \).

(iii) \( A^\alpha = \bigcap_{\alpha \geq \alpha_0} A^\alpha \), if \( \alpha_0 \uparrow 0 \).

Then there is \( u \in E^n \) (\( E^n \) denotes \( n \) dimensional fuzzy number space) such that \( [u]^\alpha = A^\alpha \) for all \( \alpha \in (0, 1] \) and \( [u]^0 = \bigcup_{\alpha \leq 1} A^\alpha \subseteq A^0 \).

**Definition 2.6.** [25] Suppose that \( A, B \in E^1 \) have \( \alpha \)-cuts \( [A]^\alpha = [A_{l, \alpha}, A_{r, \alpha}], [B]^\alpha = [B_{l, \alpha}, B_{r, \alpha}] \), with \( 0 \notin [B]^\alpha, \forall \alpha \in [0, 1] \). The \( g \)-division \( \div_g \) is the operation that calculates the fuzzy number \( C = A \div_g B \) having level cuts \( [C]^\alpha = [C_{l, \alpha}, C_{r, \alpha}] \) (here \( [A]^\alpha^{-1} = [1/A_{l, \alpha}, 1/A_{r, \alpha}] \)) defined by

\[
[C]^\alpha = [A]^\alpha \div_g [B]^\alpha \iff \begin{cases} 
(A)^\alpha = [B]^\alpha[C]^\alpha, \\
\text{or} \\
[B]^\alpha = [A]^\alpha[C]^\alpha^{-1}
\end{cases} \quad (2.2)
\]

provided that \( C \) is a proper fuzzy number (\( C_{l, \alpha} \) is nondecreasing, \( C_{r, \alpha} \) is nondecreasing, \( C_{l, 1} \leq C_{r, 1} \)).

**Remark 2.1.** According to [25], in this paper the fuzzy number is positive, if \( A \div_g B = C \in E^1 \) exists, then the following two cases are possible

Case I. if \( A_{l, \alpha}B_{r, \alpha} \leq A_{r, \alpha}B_{l, \alpha}, \forall \alpha \in [0, 1] \), then \( C_{l, \alpha} = \frac{A_{l, \alpha}}{B_{r, \alpha}} \), \( C_{r, \alpha} = \frac{A_{r, \alpha}}{B_{l, \alpha}} \).

Case II. if \( A_{l, \alpha}B_{r, \alpha} \geq A_{r, \alpha}B_{l, \alpha}, \forall \alpha \in [0, 1] \), then \( C_{l, \alpha} = \frac{A_{l, \alpha}}{B_{r, \alpha}} \), \( C_{r, \alpha} = \frac{A_{r, \alpha}}{B_{l, \alpha}} \).

The fuzzy analog of the boundedness and persistence (see [21, 22]) is as follows:

**Definition 2.7.** A sequence of positive fuzzy numbers \( (x_n) \) is persistence (resp. bounded) if there exists a positive real number \( M \) (resp. \( N \)) such that

\[
\sup\{x_n : n \subseteq [M, \infty]\} \text{ (resp. supp } x_n \subseteq (0, N)]\}, n = 1, 2, \ldots.
\]

A sequence of positive fuzzy numbers \((x_n)\) is bounded and persistence if there exist positive real numbers \( M, N > 0 \) such that

\[
\sup x_n \subseteq [M, N], n = 1, 2, \ldots.
\]

A sequence of positive fuzzy numbers \((x_n), n = 1, 2, \ldots\), is an unbounded if the norm \( \|x_n\|, n = 1, 2, \ldots \), is an unbounded sequence.

**Definition 2.8.** \( x_n \) is a positive solution of (1.1) if \((x_n)\) is a sequence of positive fuzzy numbers which satisfies (1.1). A positive fuzzy number \( x \) is called a positive equilibrium of (1.1) if

\[
x = \frac{Ax}{1 + Bx}.
\]
Let \( (x_n) \) be a sequence of positive fuzzy numbers and \( x \) is a positive fuzzy number, \( x_n \to x \) as \( n \to \infty \) if \( \lim_{n \to \infty} D(x_n, x) = 0 \).

3 Main results

3.1 Existence of solution of Beverton-Holt population model (1.1)

Firstly we study the existence of positive solutions of Beverton-Holt population model (1.1). We need the following lemma.

**Lemma 3.1.** [24] Let \( f : R^+ \times R^+ \times R^+ \to R^+ \) be continuous, \( A, B, C \) are fuzzy numbers. Then

\[
[f(A, B, C)]^\alpha = f([A]^\alpha, [B]^\alpha, [C]^\alpha), \quad \alpha \in (0, 1) \tag{3.1}
\]

**Theorem 3.1.** Let parameters \( A, B, \bar{I} \) and initial value \( x_0 \) of Beverton-Holt population model (1.1) be fuzzy numbers. Then, for any positive fuzzy number \( x_0 \), there exists a unique positive solution \( x_n \) of Beverton-Holt population model (1.1) with initial conditions \( x_0 \).

**Proof.** The proof is similar to those of Proposition 2.1 [9]. Suppose that there exists a sequence of fuzzy numbers \( (x_n) \) satisfying (1.1) with initial condition \( x_0 \). Consider the \( \alpha \)-cuts, \( \alpha \in (0, 1] \),

\[
[x_n]^\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad n = 0, 1, 2, \ldots, \quad [A]^\alpha = [A_{l,\alpha}, A_{r,\alpha}], \quad [B]^\alpha = [B_{l,\alpha}, B_{r,\alpha}], \quad \bar{I}^\alpha = [\bar{I}_{l,\alpha}, \bar{I}_{r,\alpha}] \tag{3.2}
\]

It follows from (1.1), (3.2) and Lemma 3.1 that

\[
[x_{n+1}]^\alpha = \frac{[A_{l,\alpha} L_{n,\alpha}, A_{r,\alpha} R_{n,\alpha}]}{[1_B + B_{l,\alpha} L_{n,\alpha}, 1_B + B_{r,\alpha} R_{n,\alpha}]} \tag{3.3}
\]

**Case I**

\[
[x_{n+1}]^\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[ \frac{A_{l,\alpha} L_{n,\alpha}}{1_B + B_{l,\alpha} L_{n,\alpha}}, \frac{A_{r,\alpha} R_{n,\alpha}}{1_B + B_{r,\alpha} R_{n,\alpha}} \right] \tag{3.3}
\]

**Case II**

\[
[x_{n+1}]^\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[ \frac{A_{l,\alpha} L_{n,\alpha}}{1_B + B_{l,\alpha} L_{n,\alpha}}, \frac{A_{r,\alpha} R_{n,\alpha}}{1_B + B_{r,\alpha} R_{n,\alpha}} \right] \tag{3.4}
\]

If Case I holds true, it follows that for \( n \in \{0, 1, 2, \ldots\}, \alpha \in (0, 1] \),

\[
L_{n+1,\alpha} = \frac{A_{l,\alpha} L_{n,\alpha}}{1_B + B_{l,\alpha} L_{n,\alpha}}, \quad R_{n+1,\alpha} = \frac{A_{r,\alpha} R_{n,\alpha}}{1_B + B_{r,\alpha} R_{n,\alpha}} \tag{3.5}
\]

Then it is obvious that, for any initial condition \( (L_{0,\alpha}, R_{0,\alpha}), \alpha \in (0, 1] \), there is a unique solution \( (L_{n,\alpha}, R_{n,\alpha}) \). Now we prove that \( (L_{n,\alpha}, R_{n,\alpha}), \alpha \in (0, 1] \), where \( (L_{n,\alpha}, R_{n,\alpha}) \) is the solution of system (3.5) with initial conditions \( (L_{0,\alpha}, R_{0,\alpha}) \), determines the solution \( x_n \) of (1.1) with initial conditions \( x_0 \) such that

\[
[x_n]^\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad \alpha \in (0, 1], \quad n = 0, 1, 2, \ldots \tag{3.6}
\]

For \( n = 1 \), since \( A, B, \bar{I} \) and \( x_0 \) are positive fuzzy numbers, it is easy to see that \( [L_{1,\alpha}, R_{1,\alpha}] \) is the \( \alpha \)-cuts of \( x_1 = \frac{A x_0}{1_B + B x_0} \), for any \( \alpha \in (0, 1] \), we have

\[
[L_{1,\alpha}, R_{1,\alpha}] = \left[ \frac{A_{l,\alpha} L_{0,\alpha}}{1_B + B_{l,\alpha} L_{0,\alpha}}, \frac{A_{r,\alpha} R_{0,\alpha}}{1_B + B_{r,\alpha} R_{0,\alpha}} \right] = [A]^\alpha [x_0]^\alpha \]
Working inductively, let \([L_{k,\alpha}, R_{k,\alpha}], k \geq 1\), be the \(\alpha\)-cuts of fuzzy number \(x_k\) as \([x_k]^{\alpha} = [L_{k,\alpha}, R_{k,\alpha}]\), we show that \([L_{k+1,\alpha}, R_{k+1,\alpha}]\) determines the \(\alpha\)-cuts of fuzzy number \(x_{k+1} = \frac{A_{0k}}{1+B_{x_k}}\).

According to (3.5), for \(\alpha \in (0, 1)\), we have
\[
\begin{align*}
[L_{k+1,\alpha}, R_{k+1,\alpha}] &= \left[\frac{A_{1,\alpha}L_{k,\alpha}}{1+L_{1,\alpha}+B_{1,\alpha}L_{k,\alpha}}, \frac{A_{r,\alpha}R_{k,\alpha}}{1+r,\alpha+B_{r,\alpha}R_{k,\alpha}}\right] \\
&= \left[\frac{[A]^{\alpha}[x_k]^{\alpha}}{[1]^{\alpha}+[B]^{\alpha}[x_k]^{\alpha}}\right].
\end{align*}
\] (3.7)

Then \([L_{k+1,\alpha}, R_{k+1,\alpha}]\) determines the \(\alpha\)-cuts of fuzzy number \(x_{k+1} = \frac{A_{0k}}{1+B_{x_k}}\). Then for each \(n\), \([L_{n,\alpha}, R_{n,\alpha}], \alpha \in (0, 1)\) determines the \(\alpha\)-cuts of fuzzy number \(x_n\) satisfying with (3.6).

Next we prove the uniqueness of the solution. Suppose that there exists another solution \(\pi_n\) of (1.1) with the initial condition \(x_0\). Then arguing as above, it is easy to get
\[
[\pi_n]^{\alpha} = [L_{n,\alpha}, R_{n,\alpha}], \alpha \in (0, 1], n = 0, 1, \ldots.
\] (3.8)

Then from (3.6) and (3.8) we have \([x_n]^{\alpha} = [\pi_n]^{\alpha}, \alpha \in (0, 1], n = 0, 1, 2, \ldots\), from which it follows that \(x_n = \pi_n, n = 0, 1, \ldots\). Thus the proof is completed.

If Case II holds true, the proof is similar to those of Case I. Thus the proof of Theorem 3.1 is completed.

**Remark 3.1.** From theoretical point of view, the existence of solution for fuzzy difference equation is very important and meaningful with initial condition. Therefore, in that sense, the existence of positive fuzzy solution for Bevorton-Holt population model is of vital importance and practical significance. In fact, it is clear that the positive solution of Bevorton-Holt population model with fuzzy environment is a sequence of positive fuzzy numbers describing the fuzzy uncertainty.

### 3.2 Dynamics of Bevorton-Holt population model (1.1)

To study the dynamical behavior of the positive solutions of Bevorton-Holt population model (1.1), according to Definition 2.3, we consider two cases.

First, if Case I holds true, we need the following lemma.

**Lemma 3.2** Consider the system of difference equations
\[
y_{n+1} = \frac{ay_n}{p+cy_n}, \quad z_{n+1} = \frac{bz_n}{q+dz_n}, \quad n = 0, 1, \ldots
\] (3.9)

where \(p \in (0, 1), q \in (1, +\infty), a, b, c, d \in (0, +\infty), y_0, z_0 \in (0, +\infty), \) if \(a > p, b > q, \) then the following statements are true:

(i) The system exists unique positive equilibrium \(\left(\frac{a-p}{c}, \frac{b-q}{d}\right)\) which is globally asymptotically stable.

(ii) \(y_n\) and \(z_n\) are bounded and persistent.

**Proof.** (i) Let \((\overline{y}, \overline{z})\) be equilibrium point of (3.9). It is easy to get positive equilibrium \((\overline{y}, \overline{z}) = \left(\frac{a-p}{c}, \frac{b-q}{d}\right)\). The linearized equation associated with (3.9) about equilibrium \((\overline{y}, \overline{z})\) is
\[
y_{n+1} = \frac{p}{a}y_n, \quad z_{n+1} = \frac{q}{b}z_n.
\]

Since \(a > p, b > q, \) it follows that the system is locally asymptotically stable.

On the other hand, set \(f(y) = \frac{ay}{p+cy}, g(z) = \frac{bz}{q+dz}, \) then
\[
f'(y) = \frac{ap}{(p+cy)^2} > 0, \quad g'(z) = \frac{bq}{(q+dz)^2} > 0.
\] (3.10)

Namely, the sequences \((y_n)\) and \((z_n)\) are increasing and
\[
y_n = \frac{ay_{n-1}}{p+cy_{n-1}} < \frac{a}{c}, \quad z_n = \frac{bz_{n-1}}{q+dz_{n-1}} < \frac{b}{d}.
\] (3.11)
Therefore, from (3.10) and (3.11), it follows that the limitation of \((y_n),(z_n)\) exist. Set \(\lim_{n \to \infty} y_n = y\), \(\lim_{n \to \infty} z_n = z\), integrating with (3.9), it can follows that

\[
\lim_{n \to \infty} y_n = \frac{a - p}{c}, \quad \lim_{n \to \infty} z_n = \frac{b - q}{d}.
\]

(3.12)

Therefore, it follows that the positive equilibrium \((\frac{a - p}{c}, \frac{b - q}{d})\) is globally asymptotically stable.

(ii) Set \(Y_n = \frac{1}{y_n}, Z_n = \frac{1}{z_n}\), then (3.9) can be transformed into

\[
Y_{n+1} = \frac{c}{a} + \frac{p}{a}Y_n, \quad Z_{n+1} = \frac{d}{b} + \frac{q}{b}Z_n, \quad n = 0, 1, \ldots
\]

(3.13)

It follows from (3.13) that

\[
Y_n = \frac{c}{a} + \frac{p}{a}Y_{n-1} + \frac{p^2}{a^2} + \cdots + \frac{p^n}{a^n}Y_0 = \frac{c}{a} + \frac{p^n}{a^n}Y_0 \leq \frac{c}{a - p} + Y_0 := \frac{1}{\delta}
\]

(3.14)

\[
Z_n = \frac{d}{b} + \frac{q}{b}Z_{n-1} + \frac{q^2}{b^2} + \cdots + \frac{q^n}{b^n}Z_0 = \frac{d}{b} + \frac{q^n}{b^n}Z_0 \leq \frac{d}{b - q} + Z_0 := \frac{1}{\gamma}
\]

(3.15)

From (3.11), (3.14) and (3.15), This implies

\[
\delta \leq y_n \leq \frac{a}{c}, \quad \gamma \leq z_n \leq \frac{b}{d}.
\]

This completes the proof of Lemma 3.2.

**Theorem 3.2** Consider Beverton-Holt population model (1.1) with fuzzy uncertainty parameters and initial condition. If

\[
0 < \tilde{A}_{t,\alpha} \leq A_{t,\alpha}, \quad \bar{A}_{r,\alpha} \leq A_{r,\alpha}, \quad \alpha \in (0, 1],
\]

(3.16)

then the following statements are true.

(i) Every positive solution \(x_n\) of Beverton-Holt population model (1.1) is bounded and persistent.

(ii) Every positive solution \(x_n\) of Beverton-Holt population model (1.1) tends to the positive equilibrium point \(x\) as \(n \to \infty\).

**Proof.** (i) Since \(A, \tilde{A}, \bar{B}\) and the initial value \(x_0\) are positive fuzzy numbers, there exist positive real numbers \(M_A, N_A, M_B, N_B, M_0, N_0, P, Q\), such that, for all \(\alpha \in (0, 1]\),

\[
[A_{t,\alpha}, A_{r,\alpha}] \subset [M_A, N_A], \quad [B_{t,\alpha}, B_{r,\alpha}] \subset [M_B, N_B], \quad [L_{0,\alpha}, R_{0,\alpha}] \subset [M_0, N_0], \quad [\bar{A}_{t,\alpha}, \bar{A}_{r,\alpha}] \subset [P, Q].
\]

(3.17)

Let \(x_n\) be a positive solution of Beverton-Holt population model (1.1), from (3.16), (3.17) and Lemma 3.2, we get

\[
L_{n,\alpha} > \frac{(A_{t,\alpha} - \bar{A}_{t,\alpha})L_{0,\alpha}}{B_{t,\alpha}L_{0,\alpha} + A_{t,\alpha} - \bar{A}_{t,\alpha}} \geq \frac{(M_A - Q)M_0}{N_BN_0 + N_A - Q} := M, \quad R_{n,\alpha} < \frac{A_{r,\alpha}}{B_{r,\alpha}} < \frac{N_A}{M_B} := N
\]

(3.18)

From which, we get for \(n \geq 1, \cup_{\alpha \in (0, 1]} [L_{n,\alpha}, R_{n,\alpha}] \subset [M, N]\), and so \(\cup_{\alpha \in (0, 1]} [L_{n,\alpha}, R_{n,\alpha}] \subset [M, N]\). Thus the positive solution is bounded and persistent.

(ii) Suppose that there exists a fuzzy number \(x\) such that

\[
x = \frac{Ax}{1 + Bx}, \quad [x] = [L_{\alpha}, R_{\alpha}], \quad \alpha \in (0, 1].
\]

(3.19)
where \( L_\alpha, R_\alpha \geq 0 \). Then from (3.19) we can prove that
\[
L_\alpha = \frac{A_{t,\alpha} L_\alpha}{l_{t,\alpha} + B_{t,\alpha} L_\alpha}, \quad R_\alpha = \frac{A_{r,\alpha} R_\alpha}{l_{r,\alpha} + B_{r,\alpha} R_\alpha}.
\] (3.20)

Hence from (3.20), we have
\[
L_\alpha = \frac{A_{t,\alpha} - \tilde{l}_{t,\alpha}}{B_{t,\alpha}}, \quad R_\alpha = \frac{A_{r,\alpha} - \tilde{l}_{r,\alpha}}{B_{r,\alpha}}.
\] (3.21)

Let \( x_n \) be a positive solution of Beverton-Holt population model (1.1). Since (3.16) holds true, we can apply Lemma 3.2 to system (3.5), and so we have
\[
\lim_{n \to \infty} L_{\alpha,\alpha} = L_\alpha, \quad \lim_{n \to \infty} R_{\alpha,\alpha} = R_\alpha,
\] (3.22)

Therefore from (3.22) we have
\[
\lim_{n \to \infty} D(x_n, x) = \lim_{n \to \infty} \sup_{a \in [0,1]} \{\max\{|L_{\alpha,\alpha} - L_\alpha|, |R_{\alpha,\alpha} - R_\alpha|\}\} = 0.
\]

This completes the proof of the Theorem.

Secondly, if Case II holds true, it follows that for \( n \in \{0, 1, 2, \cdots\} \), \( \alpha \in (0, 1] \)
\[
L_{n+1,\alpha} = \frac{A_{r,\alpha} R_{n,\alpha}}{l_{r,\alpha} + B_{r,\alpha} R_{n,\alpha}}, \quad R_{n+1,\alpha} = \frac{A_{t,\alpha} L_{n,\alpha}}{l_{t,\alpha} + B_{t,\alpha} L_{n,\alpha}}
\] (3.23)

We need the following lemmas.

**Lemma 3.3.** Consider the system of difference equations
\[
y_{n+1} = \frac{b y_n}{q + d z_n}, \quad z_{n+1} = \frac{a y_n}{c + b z_n}, \quad n = 0, 1, \cdots,
\] (3.24)
where \( p \in (0, 1), q \in (1, +\infty), a, b, p, q, c, d, y_0, z_0 \in (0, +\infty) \). If
\[
a > p, \quad b > q,
\] (3.25)
then the following statements are true.

(i) The solution of (3.24) is bounded and persistent.

(ii) Furthermore suppose that
\[
\frac{a - p}{c} < \frac{b - q}{d},
\] (3.26)
then there exists unique positive equilibrium point \( (\frac{ab - pq}{qc + ad}, \frac{ab - pq}{pd + bc}) \) which is globally asymptotically stable.

**Proof.** (i) It is clear from (3.24) that
\[
y_n \leq \frac{b}{d}, \quad z_n \leq \frac{a}{c}.
\] (3.27)

Setting \( Y_n = \frac{1}{y_n}, \quad Z_n = \frac{1}{z_n} \), then (3.24) can be transformed to
\[
Y_n = \frac{d}{b} + \frac{q}{b} Z_{n-1}, \quad Z_n = \frac{c}{a} + \frac{p}{a} Y_{n-1}.
\] (3.28)

Working inductively, for \( n - 2k \geq 0 \), it can conclude that
\[
Y_n = \frac{d}{b} + \frac{q}{b} \left[ \frac{c}{a} + \frac{p}{a} Y_{n-2} \right] = \frac{d}{b} + \frac{q c}{a b} + \frac{p q}{a b} Y_{n-2}
\]
\[
= \frac{d}{b} + \frac{q c}{a b} + \frac{p q d}{a b^2} + \frac{p q^2}{a b^2} Z_{n-3} = \frac{d}{b} + \frac{q c}{a b} + \frac{p q d}{a b^2} + \frac{p q^2 c}{a^2 b^2} + \frac{p^2 q^2}{a^2 b^2} Y_{n-4}
\]
\[
= \cdots = \left[ \frac{d}{b} - \frac{pq}{ab} \right] \left[ 1 - \left( \frac{pq}{ab} \right)^k \right] + \frac{q c}{a b} \left[ 1 - \left( \frac{pq}{ab} \right)^k \right] + \left( \frac{pq}{ab} \right)^k \frac{c}{a} + \left( \frac{pq}{ab} \right)^k Y_{n-2k}
\]
\[
\leq \frac{d}{b} + \frac{q c}{a b} + Y_0 = \frac{b c + p d}{a b - pq} + Y_0 \quad \text{where} \quad Y_0 := \frac{1}{\xi}.
\] (3.29)
\[ Z_n = \frac{c + p}{a} \left( \frac{d}{b} + \frac{q}{b} Z_{n-2} \right) = \frac{c}{a} + \frac{p d}{a b} + \frac{p q c}{a^2 b^2} + \frac{p^2 q d}{a^2 b^2} + \frac{p^2 q^2}{a^2 b^2} Z_{n-4} = \cdots = \frac{c}{1 - \left( \frac{p q}{a b} \right)^k} + \frac{p d}{a b} \left( 1 - \left( \frac{p q}{a b} \right)^k \right) + \left( \frac{p q}{a b} \right)^k Z_{n-2k} \leq \frac{c}{1 - \frac{p q}{a b}} + Z_0 = \frac{b c + p d}{a b - p q} + Z_0 := \frac{1}{\rho} \]

(3.30)

It follows from (3.27), (3.29) and (3.30) that

\[ \xi \leq y_n \leq \frac{b}{a}, \quad \rho \leq z_n \leq \frac{a}{c}, \quad \text{for} \quad n > 0. \]

(3.31)

This completes the proof of (i).

(ii) Let \((\bar{y}, \bar{z})\) be an equilibrium point of (3.24). It is easy to obtain that \((\bar{y}, \bar{z}) = \left( \frac{ab - p q}{qc + ad}, \frac{ab - p q}{pd + bc} \right)\). The linearized equation of system (3.24) about the equilibrium point \((\bar{y}, \bar{z})\) is

\[ \Psi_{n+1} = G \Psi_n. \]

(3.32)

where \(\Psi_n = (y_n, z_n)^T\) and

\[ G = \begin{pmatrix} 0 & \frac{q b}{(p + c \bar{y})^2} \\ \frac{p q a b}{(p + c \bar{y})^2 (q + d \bar{z})^2} & 0 \end{pmatrix} \]

Thus the characteristic equation of (3.32) is

\[ \lambda^2 - \frac{p q a b}{(p + c \bar{y})^2 (q + d \bar{z})^2} = 0. \]

Since (3.25) and (3.26) hold true, it is easy to obtain that the root of characteristic equation \(|\lambda| < 1\). From [26, 27], thus the unique positive equilibrium point \((\bar{y}, \bar{z})\) is locally asymptotically stable.

Next, let \((y_n, z_n)\) be an arbitrary positive solution of (3.24). From (3.29) and (3.30), we have \(\{Y_n\}, \{Z_n\}\) are monotone increasing and have a upper bound. Namely, \(\{y_n\}, \{z_n\}\) are monotone decreasing and have lower bound. So, set \(\lim_{n \to \infty} y_n = y, \lim_{n \to \infty} z_n = z\). From (3.24), it can follow that

\[ \lim_{n \to \infty} y_n = y = \bar{y} = \frac{ab - pq}{qc + ad}, \quad \lim_{n \to \infty} z_n = z = \bar{z} = \frac{ab - pq}{pd + bc}. \]

Therefore it implies that the unique positive equilibrium point \((\bar{y}, \bar{z})\) is globally asymptotically stable.

**Theorem 3.3.** Suppose that parameters \(A, B, \bar{1}\) and initial condition \(x_0\) of Beverton-Holt population model (1.1) are fuzzy numbers. If (3.16) and the following condition are satisfied

\[ \frac{B_1}{B_{r,\alpha}} \leq \frac{A_{l,\alpha} - \bar{1}_{l,\alpha}}{A_{r,\alpha} - \bar{1}_{r,\alpha}}, \quad \forall \alpha \in (0, 1]. \]

(3.33)

Then the following statements are true

(i) Every positive solution \(x_n\) of Beverton-Holt population model (1.1) is bounded and persistence.

(ii) Every positive solution \(x_n\) of Beverton-Holt population model (1.1) tends to the positive equilibrium point \(x\) as \(n \to +\infty\).

**Proof.** (i) The proof is similar to those of Theorem 3.2. Let \(x_n\) be a positive solution of (1.1), from (3.16), (3.17) and Lemma 3.3, we get

\[ L_{n,\alpha} \geq \frac{L_{0,\alpha}(A_{l,\alpha}A_{r,\alpha} - \bar{1}_{l,\alpha} \bar{1}_{r,\alpha})}{L_{0,\alpha}(A_{r,\alpha}B_{l,\alpha} + \bar{1}_{l,\alpha}B_{r,\alpha}) + (A_{l,\alpha}A_{r,\alpha} - \bar{1}_{l,\alpha} \bar{1}_{r,\alpha})} \geq \frac{M_0 \left( M_0^2 - Q^2 \right)}{N_0(N_A N_B + Q N_B) + (N_A^2 - P^2)} =: K \]

(3.34)
\[ R_{n,\alpha} \leq \frac{A_{t,\alpha}}{B_{t,\alpha}} \leq \frac{N_A}{M_B} =: L \]  

(3.35)

From (3.34) and (3.35), we get, for \( n \geq 1 \), \( \bigcup_{\alpha \in (0, 1]} [L_{n,\alpha}, R_{n,\alpha}] \subseteq [K, L] \). And so

\[ \bigcup_{\alpha \in (0, 1]} [L_{n,\alpha}, R_{n,\alpha}] \subseteq [K, L]. \]

Thus the positive solution \( x_n \) of (1.1) is bounded and persistent.

(ii) Suppose that there exists a fuzzy number \( x \) such that (3.19) is satisfied. Then from (3.19) and Case II, we have

\[ L_\alpha = \frac{A_{t,\alpha} R_\alpha}{1 + B_{t,\alpha} R_\alpha}, \quad R_\alpha = \frac{A_{l,\alpha} L_\alpha}{1 + B_{l,\alpha} L_\alpha}. \]  

(3.36)

It follows from (3.36) that

\[ L_\alpha = \frac{A_{t,\alpha} A_{r,\alpha} - \bar{l}_{t,\alpha} \bar{r}_{t,\alpha}}{1 + B_{t,\alpha} A_{r,\alpha}}, \quad R_\alpha = \frac{A_{l,\alpha} A_{r,\alpha} - \bar{l}_{l,\alpha} \bar{r}_{l,\alpha}}{1 + B_{l,\alpha} A_{r,\alpha}}. \]  

(3.37)

Let \( x_n \) be a positive solution of (1.1) such that Case II holds. Namely,

\[ L_{n+1,\alpha} = \frac{A_{t,\alpha} R_{n,\alpha}}{1 + B_{t,\alpha} R_{n,\alpha}}, \quad R_{n+1,\alpha} = \frac{A_{l,\alpha} L_{n,\alpha}}{1 + B_{l,\alpha} L_{n,\alpha}}. \]  

(3.38)

Since (3.38) is satisfied, we can apply Lemma 3.3 to system (3.38) and so we have

\[ \lim_{n \to \infty} L_{n,\alpha} = L_\alpha, \quad \lim_{n \to \infty} R_{n,\alpha} = R_\alpha. \]  

(3.39)

Therefore from (3.39) we have

\[ \lim_{n \to \infty} D(x_n, x) = \lim_{n \to \infty} \sup_{\alpha \in (0, 1]} \{ \max\{|L_{n,\alpha} - L_\alpha|, |R_{n,\alpha} - R_\alpha|\} \} = 0. \]

This completes the proof of Theorem 3.3.

**Remark 3.2.** In population dynamical model, the parameters of model derived from statistic data with vagueness or uncertainty. It corresponds to reality to use fuzzy parameters in population dynamical model. In contrast with classic population model, the solution of fuzzy population model is within a range of value (approximate value), which are taken into account fuzzy uncertainties. Furthermore the global asymptotic behaviour of discrete Beverton-Holt population model are obtained in fuzzy context.

### 4 Numerical examples

**Example 4.1** Consider the following fuzzy discrete time Beverton-Holt population model

\[ x_{n+1} = \frac{A x_n}{1 + B x_n}, \quad n = 0, 1, \cdots, \]  

(4.1)

we take \( A, B, \bar{1} \) and the initial values \( x_0 \) such that

\[
A(x) = \begin{cases} 
\frac{3}{2} x - 1, & 0.8 \leq x \leq 1.6 \\
-\frac{3}{2} x + 5, & 1.6 \leq x \leq 2 
\end{cases} \quad \bar{1}(x) = \begin{cases} 
2x - 1, & 0.5 \leq x \leq 1 \\
5x + 6, & 1 \leq x \leq 1.2 
\end{cases}
\]  

(4.2)

\[
B(x) = \begin{cases} 
10x - 4, & 0.4 \leq x \leq 0.5 \\
-10x + 6, & 0.5 \leq x \leq 0.6 
\end{cases} \quad x_0(x) = \begin{cases} 
x - 3, & 3 \leq x \leq 4 \\
x - 5, & 4 \leq x \leq 5 
\end{cases}
\]  

(4.3)
From (4.2), we get
\[ [A]^\alpha = \left[ 0.8 + \frac{4}{5} \alpha, 2 - \frac{2}{5} \alpha \right], \quad [\tilde{1}]^\alpha = \left[ 0.5 + \frac{1}{2} \alpha, 1.2 - \frac{1}{5} \alpha \right], \; \alpha \in (0, 1). \] (4.4)

From (4.3) we get
\[ [B]^\alpha = \left[ 0.4 + \frac{1}{10} \alpha, 0.6 - \frac{1}{10} \alpha \right], \quad [x_0]^\alpha = [3 + \alpha, 5 - \alpha], \; \alpha \in (0, 1). \] (4.5)

Therefore, it follows that
\[ \bigcup_{\alpha \in (0, 1)} [A]^\alpha = [0.8, 2], \quad \bigcup_{\alpha \in (0, 1)} [\tilde{1}]^\alpha = [0.5, 1.2], \quad \bigcup_{\alpha \in (0, 1)} [B]^\alpha = [0.4, 0.6], \quad \bigcup_{\alpha \in (0, 1)} [x_0]^\alpha = [3, 5]. \] (4.6)

From (4.1), it results in a coupled system of difference equations with parameter \( \alpha \),
\[ L_{n+1,\alpha} = \frac{A_{l,\alpha} L_{n,\alpha}}{I_{l,\alpha} + B_{l,\alpha} L_{n,\alpha}}, \quad R_{n+1,\alpha} = \frac{A_{r,\alpha} R_{n,\alpha}}{I_{r,\alpha} + B_{r,\alpha} R_{n,\alpha}}, \; \alpha \in (0, 1). \] (4.7)

Therefore, \( A_{l,\alpha} \geq \tilde{1}_{l,\alpha} > 0, A_{r,\alpha} \geq \tilde{1}_{r,\alpha}, \forall \alpha \in (0, 1), \) and initial values \( x_0 \) are positive fuzzy numbers, so from Theorem 3.2, we have that every positive solution \( x_n \) of Eq.(4.1) is bounded and persistence. In addition, from Theorem 3.2, Eq.(4.1) has a unique positive equilibrium \( \bar{x} = (0.75, 1.2, 1.333) \). Moreover every positive solution \( x_n \) of Eq.(4.1) converges the unique equilibrium \( \bar{x} \) with respect to \( D \) as \( n \to \infty \). (see Fig.1-Fig.3)

Fig.1. The Dynamics of system (4.7).

Fig.2. The solution of system (4.7) at \( \alpha = 0 \) and \( \alpha = 0.25 \).
Example 4.2 Consider the following fuzzy discrete time Beverton-Holt population model (4.1).

where \( A, \tilde{1}, B \) and the initial values \( x_0 \) are satisfied

\[
A(x) = \begin{cases}
\frac{10}{3}x - 3, & 0.9 \leq x \leq 1.2 \\
-\frac{10}{3}x + 5, & 1.2 \leq x \leq 1.5
\end{cases}, \quad \tilde{1}(x) = \begin{cases}
\frac{10}{3}x - \frac{7}{3}, & 0.7 \leq x \leq 1 \\
-\frac{10}{3}x + \frac{13}{3}, & 1 \leq x \leq 1.3
\end{cases}
\]

(4.8)

\[
B(x) = \begin{cases}
5x - 2, & 0.4 \leq x \leq 0.6 \\
-5x + 4, & 0.6 \leq x \leq 0.8
\end{cases}, \quad x_0(x) = \begin{cases}
\frac{5}{2}x - 1.5, & 0.6 \leq x \leq 1 \\
-\frac{5}{2}x + 3.5, & 1 \leq x \leq 1.4
\end{cases}
\]

(4.9)

From (4.8), we get

\[
[A]^\alpha = \left[0.9 + \frac{3}{10}\alpha, 1.5 - \frac{3}{10}\alpha\right], \quad [\tilde{1}]^\alpha = \left[0.7 + \frac{3}{10}\alpha, 1.3 - \frac{3}{10}\alpha\right], \quad \alpha \in (0,1].
\]

(4.10)

From (4.9), we get

\[
[B]^\alpha = \left[0.4 + \frac{1}{5}\alpha, 0.8 - \frac{1}{5}\alpha\right], \quad [x_0]^\alpha = \left[0.6 + \frac{2}{5}\alpha, 1.4 - \frac{2}{5}\alpha\right], \quad \alpha \in (0,1].
\]

(4.11)

Therefore, it follows that

\[
\bigcup_{\alpha \in (0,1]} [A]^\alpha = [0.9, 1.5], \quad \bigcup_{\alpha \in (0,1]} [\tilde{1}]^\alpha = [0.7, 1.3], \quad \bigcup_{\alpha \in (0,1]} [B]^\alpha = [0.4, 0.8], \quad \bigcup_{\alpha \in (0,1]} [x_0]^\alpha = [0.6, 1.4].
\]

(4.12)

From (4.1), it results in a coupled system of difference equation with parameter \( \alpha \),

\[
L_{n+1,\alpha} = \frac{A_{r,\alpha}R_{n,\alpha}}{I_{r,\alpha} + B_{r,\alpha}R_{n,\alpha}}, \quad R_{n+1,\alpha} = \frac{A_{l,\alpha}L_{n,\alpha}}{I_{l,\alpha} + B_{l,\alpha}L_{n,\alpha}}, \quad \alpha \in (0,1].
\]

(4.13)

It is clear that (3.33) is satisfied and initial values \( x_0 \) are positive fuzzy numbers, so from Theorem 3.3, Eq.(4.1) has a unique positive equilibrium \( x = (0.333, 0.3548, 0.3793) \). Moreover every positive solution \( x_n \) of Eq.(4.1) converges the unique equilibrium \( \bar{x} \) with respect to \( D \) as \( n \to \infty \). (see Fig.4-Fig.6)
5 Conclusion

In this work, according to a generalization of division (g-division) of fuzzy number, we study the fuzzy discrete time Beverton-Holt population model \( x_{n+1} = \frac{Ax_n}{1+Bx_n} \). The existence of positive solution and qualitative behavior to (1.1) are investigated. The main results are as follows
(1) Under Case I, the positive solution is bounded and persists if $A_{l, \alpha} > \bar{A}_{l, \alpha}, A_{r, \alpha} > \bar{A}_{r, \alpha}, \alpha \in (0, 1]$. Every positive solution $x_n$ tends to the unique equilibrium $x$ as $n \to \infty$.

(2) Under Case II, the positive solution is bounded and persists if $A_{l, \alpha} > \bar{A}_{l, \alpha}, A_{r, \alpha} > \bar{A}_{r, \alpha}, B_{l, \alpha}, B_{r, \alpha} \leq A_{l, \alpha} - \bar{A}_{l, \alpha}, A_{r, \alpha} - \bar{A}_{r, \alpha}, \alpha \in (0, 1]$. Every positive solution $x_n$ tends to the unique equilibrium $x$ as $n \to \infty$.

Competing interests
The authors declare that they have no competing interests.

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