AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 1(x): xxx–xxx
DOI:10.3934/Math.2018.x.xxx
Received: July 2017
Accepted: September 2017
Published: December 2017

Research article

Wave Equations & Energy

William O. Bray[1*] **Ellen Hunter**[2]

[1] Department of Mathematics, Missouri State University, 901 S. National, Springfield, MO 65897, USA

[2] Department of Mathematics, Missouri State University, 901 S. National, Springfield, MO 65897, USA

* **Correspondence:** wbray@missouristate.edu

Abstract: The focus of this work is apply Fourier analytic methods based on Parseval's equality to the computation of kinetic and potential energy of solutions of initial boundary value problems for general wave type equations on a finite interval. As a consequence, an energy equipartion principle for the solution is obtained. Within our methods are some new results regarding eigenfunction expansions arising from regular Sturm-Liouville problems in Sobolev spaces.

Keywords: wave equation, Sturm-Liouville problem, Sobolev space, energy conservation, energy equipartition
Mathematics Subject Classification: 35L05, 34B24

1. Introduction

The focus of this paper is to apply Fourier analytic methods in the context of eigenfunction expansions in Sobolev spaces to compute the kinetic and potential energy for solutions of initial boundary value problems of the form

$$\begin{aligned} &\text{PDE: } \rho u_{tt} = (ku)_x,\ a < x < b,\ t > 0 \\ &\text{BC: } \begin{cases} \sin\theta_1 u_x(a,t) - \cos\theta_1 u(a,t) = 0 \\ \sin\theta_2 u_x(b,t) + \cos\theta_2 u(b,t) = 0 \end{cases} \\ &\text{IC: } \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases} \end{aligned} \tag{1.1}$$

Throughout this paper the above partial differential equation will be called the wave equation. Further, the coefficients are assumed to satisfy the following classical [6] conditions.

$$\begin{cases} \text{(i) } k = k(x) \in C^1[a,b],\ \rho = \rho(x) \in C[a,b], \\ \qquad k(x), \rho(x) > 0 \text{ on } [a,b] \\ \text{(ii) } 0 \le \theta_i \le \pi/2,\ i = 1,2 \end{cases} \tag{1.2}$$

Applying separation of variables to the wave equation, $u(x,t) = X(x)T(t)$, leads to the boundary value problem:

$$\begin{aligned} &\text{ODE: } (kX')' + \mu\rho X = 0,\ a < x < b \\ &\text{BC: } \begin{cases} \sin\theta_1 X'(a) - \cos\theta_1 X(a) = 0 \\ \sin\theta_2 X'(b) + \cos\theta_2 X(b) = 0 \end{cases}. \end{aligned} \tag{1.3}$$

The boundary conditions are called Dirichlet (D) if $\theta_1 = \theta_2 = 0$, Neumann (N) if $\theta_1 = \theta_2 = \pi/2$, mixed (M) if either $\theta_1 = 0$ and $\theta_2 = \pi/2$ or $\theta_1 = \pi/2$ and $\theta_2 = 0$, and Robin (R) if $0 < \theta_k < \pi/2$ for at least one $k = 1, 2$. It is easy to see that the boundary value problem (1.3) has eigenvalue $\mu = 0$ if and only if the boundary conditions are Neumann. In the latter case we set $\mu_0 = 0$ and $\Lambda = \mathbb{N}\cup\{0\}$, in all other cases, set $\Lambda = \mathbb{N}$. Under the conditions (1.2) [6], the boundary value problem has an increasing sequence $\{\mu_n\}_{n\in\Lambda}$ on non-negative eigenvalues, $\mu_n \to \infty$, with corresponding sequence of real valued eigenfunctions $X = \{X_n(x)\}_{n\in\Lambda} \subset C^2[a,b]$ that form a complete orthogonal system for the Hilbert space $L^2_\rho[a,b]$, the space of all real valued measurable functions square integrable with respect to the weight $\rho(x)$, the inner product and norm given by

$$(f,g)_\rho = \int_a^b f(x)g(x)\rho(x)\,dx,\ \ \|f\|^2_{2,\rho} = \int_a^b f^2(x)\rho(x)\,dx, \tag{1.4}$$

respectively. (In settings where the weight $\rho = 1$, the subscript will be dropped. Similar notation is used for other weights.)

Given $f \in L^2_\rho[a,b]$, its Fourier expansion with respect to $\{X_n\}$ is given by

$$\begin{aligned} f &\sim \sum_{n\in\Lambda} \widehat{f_X}(n) X_n(x), \text{ where} \\ \widehat{f_X}(n) &= \frac{1}{\|X_n\|^2_{2,\rho}} \int_a^b f(x)\, X_n(x)\,\rho(x)dx,\ n \in \Lambda. \end{aligned} \tag{1.5}$$

As it is a primary tool and for comparative purposes, Parseval's theorem for this expansion is stated here in the following form [6].

Theorem 1.1 (Parseval)**.** *If $f \in L^2_\rho[a,b]$, then*

$$\|f\|^2_{2,\rho} = \sum_{n\in\Lambda} \widehat{f_X}(n)^2 \|X_n\|^2_{2,\rho}. \tag{1.6}$$

Moreover, for $f, g \in L^2_\rho[a,b]$, the dual form of the above formula is:

$$(f,g)_\rho = \sum_{n\in\Lambda} \widehat{f_X}(n)\widehat{g_X}(n)\|X_n\|^2_{2,\rho}.$$

Conversely, if $\{c_n\}_{n\in\Lambda} \subset \mathbb{R}$ such that $\sum_{n\in\Lambda} c_n^2 \|X_n\|^2_{2,\rho} < \infty$, then there is $f \in L^2_\rho[a,b]$ such that $c_n = \widehat{f_X}(n)$, $n \in \Lambda$.

Returning to the perspective of separation of variables, the Fourier analytic form for the solution of IBVP (1.1) becomes:

$$\sum_{n\in\Lambda} \widehat{f_X}(n)\, \cos\, \sqrt{\mu_n} t\, X_n(x). \tag{1.7}$$

By Parseval's theorem, there is a function $u(\cdot,t) \in L^2_\rho[a,b]$ such that

$$\widehat{u(\cdot,t)}_X(n) = \widehat{f_X}(n) \cos\, \sqrt{\mu_n} t,\ n \in \Lambda$$

and further,

$$\sup_{t\geq 0} \|u_N(\cdot,t) - u(\cdot,t)\|^2_{2,\rho} \leq \sum_{n=N+1}^{\infty} \widehat{f_X}(n)^2 \|X_n\|^2_{2,\rho} \to 0\, (N \to \infty),$$

where $u_N(x,t)$ denotes the N^{th} partial sum of the series in (1.7). It is then easy to show that $u(x,t)$ is a weak solution of the wave equation, i.e.,

$$\int_0^\infty \int_a^b u(x,t) \left[\rho(x)\phi_{tt}(x,t) - (k(x)\phi_x(x,t))_x\right] dxdt = 0,$$

for all $\phi \in C_c^\infty((a,b)\times(0,\infty))$.

In order to fix energy forms for all boundary conditions, recall the energy method applied to the wave equation. Suppose $u(x,t)$ is a smooth solution of the wave equation, multiply the equation by u_t, integrate over the spacial interval $[a,b]$, apply integration by parts on the right hand side, then upon rearrangement of terms:

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t(\cdot,t)\|^2_{2,\rho} + \frac{1}{2}\|u_x(\cdot,t)\|^2_{2,k}\right] = \text{B.T.}, \tag{1.8}$$

where

$$\text{B.T.} = k(b)u_x(b,t)u_t(b,t) - k(a)u_x(a,t)u_t(a,t). \tag{1.9}$$

Independent of boundary conditions, the first term in brackets in (1.8) is the total kinetic energy term and will be denoted KE(t):

$$\text{KE}(t) = \frac{1}{2}\|u_t(\cdot,t)\|^2_{2,\rho}. \tag{1.10}$$

If the boundary conditions are of type (D), (N), or (M), then B.T. $= 0$ and (1.8) represents energy conservation where the second term in brackets is the total potential energy, denoted PE(t). In the case of Robin boundary conditions, the total potential energy must be modified due to the exchange of potential energy at the boundary. Indeed, assuming $0 < \theta_k < \pi/2$, for $k = 1,2$, substituting from the boundary conditions gives

$$\text{B.T.} = -\frac{d}{dt}\frac{1}{2}\left[k(b)\cot\theta_2 u^2(b,t) + k(a)\cot\theta_1 u^2(a,t)\right]$$

and hence the total potential energy terms is:

$$\text{PE}(t) = \frac{1}{2}\|u_x(\cdot,t)\|^2_{2,k} + \frac{1}{2}\left[k(b)\cot\theta_2 u^2(b,t) + k(a)\cot\theta_1 u^2(a,t)\right]. \tag{1.11}$$

Remark 1.1. *It should be clear to the reader and is to be understood in the sequel, that if $\theta_1 = 0$ or $\theta_2 = 0$, then the corresponding term on the right hand side is dropped. With this convention, (1.11) provides the formula for potential energy for any of the boundary conditions under consideration.*

1.1. A Model Example

Consider the following classical vibrating string initial value problem (IBVP)

$$\begin{aligned} &\text{PDE: } u_{tt} = u_{xx},\ 0 < x < \pi,\ t > 0 \\ &\text{BC: } u(0,t) = u(\pi,t) = 0 \\ &\text{IC: } \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases} \end{aligned} \tag{1.12}$$

where $u = u(x,t)$. If $f \in L^2[0,\pi]$, separation of variables provides a weak solution of the above PDE given by

$$u(x,t) = \sum_{n=1}^{\infty} \widehat{f_s}(n)\ \cos nt\ \sin nx, \tag{1.13}$$

where the Fourier sine coefficients are given by

$$\widehat{f_s}(n) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\, dx,\ n = 1, 2, \ldots.$$

Parseval's equality in this case has form $\|f\|_2^2 = \frac{\pi}{2} \sum_{n=1}^{\infty} \widehat{f_s}(n)^2$.

The kinetic and potential energies respectively, are given by:

$$\mathrm{KE}(t) = \frac{1}{2} \int_0^{\pi} u_t^2(x,t)\, dx,\ \ \mathrm{PE}(t) = \frac{1}{2} \int_0^{\pi} u_x^2(x,t)\, dx,$$

for which a stronger hypothesis on the intial data is needed. Thinking in terms of differentiating the series (1.13) term by term and Parseval's equality, it is natural to assume

$$\sum_{n=1}^{\infty} n^2 \widehat{f_s}(n)^2 < \infty. \tag{1.14}$$

This hypothesis has significant consequences and are listed below.

1. $\sum_n |\widehat{f_s}(n)| < \infty$. Hence, the Fourier sine series of f,

$$f \sim \sum_{n=1}^{\infty} \widehat{f_s}(n) \sin nx, \tag{1.15}$$

   converges uniformly to f on $[0,\pi]$, $f \in C[0,\pi]$, and

$$f(0) = f(\pi) = 0. \tag{1.16}$$

2. Again as a consequence of Parseval's equality, the weak derivative f' of f exists, $f' \in L^2[0,\pi]$, and is given in $L^2[0,\pi]$ by the term by term derivative of the series (1.15). Furthermore, the Fourier cosine coefficients of f',

$$\widehat{f'}_c(n) = \frac{2}{\pi} \int_0^{\pi} f'(x) \cos nx\, dx,$$

   are given by

$$\widehat{f'}_c(n) = n\, \widehat{f_s}(n),\ n = 1, 2, \ldots,\ \widehat{f'}_c(0) = 0. \tag{1.17}$$

3. The series (1.13) converges uniformly on $[0,\pi] \times [0,\infty)$, so the weak solution $u(x,t)$ satisfies the boundary conditions, and further, the weak derivatives $u_t(x,t)$ and $u_x(x,t)$ are in $L^2[0,\pi]$ for all $t > 0$. Consequently, the kinetic and potential energies can be computed using Parseval's equality:

$$\mathrm{KE}(t) = \frac{\pi}{4} \sum_{n=1}^{\infty} n^2 \widehat{f_s}(n)^2 \sin^2 nt, \ \mathrm{PE}(t) = \frac{\pi}{4} \sum_{n=1}^{\infty} n^2 \widehat{f_s}(n)^2 \cos^2 nt \tag{1.18}$$

and the energy conservation reads

$$\mathrm{KE}(t) + \mathrm{PE}(t) = \frac{\pi}{4} \sum_{n=1}^{\infty} n^2 \widehat{f_s}(n)^2 = \frac{1}{2} \|f'\|_2^2 = E.$$

4. From (1.18), the kinetic and potential energy can be expressed in the form

$$\mathrm{KE}(t) = \frac{E}{2} - g(t), \ \mathrm{PE}(t) = \frac{E}{2} + g(t), \tag{1.19}$$

where $g(t)$ is a continuous π−periodic function with mean value over a period equal to zero. Thus, (1.19) represents an energy equipartition principle for the vibrating string: averaging the kinetic and potential energy over a time cycle, the resulting values are both one-half the total energy E. This result seems to have been first noticed in [5].

Item 2 above implies that $f \in W^{1,2}[0,\pi]$, the Sobolev space of all $g \in L^2[0,\pi]$ whose weak derivatives $g' \in L^2[0,\pi]$. Two things are hidden in items 1 and 2 above. The first is the fact that within the space $W^{1,2}[0,\pi]$, statements (1.14), (1.16), and (1.17) are equivalent. The space $W^{1,2}[0,\pi]$ is a Hilbert space with natural inner product and norm given by

$$(f,g)_{W^{1,2}} = (f,g) + (f',g'), \ \|f\|_{W^{1,2}}^2 = \|f\|_2^2 + \|f'\|_2^2.$$

The second hidden observation is that the completeness of the orthogonal systems $\{\sin nx\}_1^\infty$ and $\{\cos nx\}_0^\infty$ in $L^2[0,\pi]$ manifests in the Sobolev setting as the completion of the orthogonal set $\{\sin nx\}_1^\infty$ in the Hilbert space $W_d^{1,2}[0,\pi]$ (inner product and norm inherited from $W^{1,2}[0,\pi]$) of all functions $f \in W^{1,2}[0,\pi]$ that satisfy (1.16). These facts are consequences of more general results in the sequel and lie at the heart of extending items 1-4 to more general initial boundary value problems.

1.2. Overview of the Main Results

The Sobolev space $W^{1,2}[a,b]$ is defined as the class of all $f \in L_\rho^2[a,b]$ whose weak derivative $f' \in L_k^2[a,b]$. Functions in $W^{1,2}[a,b]$ that agree except on a set of measure zero are identified. Imposing the inner product

$$(f,g)_{W^{1,2}} = (f,g)_\rho + (f',g')_k,$$

and induced norm

$$\|f\|_{W^{1,2}}^2 = \|f\|_{2,\rho}^2 + \|f'\|_{2,k}^2, \tag{1.20}$$

$W^{1,2}[a,b]$ is a Hilbert space. Notice that the norm is equivalent to the usual norm (one without weights) due to the conditions on the functions ρ and k. The following two fundamental results on Sobolev spaces will be of use in the sequel [9, 10].

- The Sobolev embedding theorem states that $W^{1,2}[a,b] \subset C[a,b]$ and the Sobolev inequality is:

$$\|f\|_\infty = \max_{x\in[a,b]} |f(x)| \leq C\,\|f\|_{W^{1,2}},\ f \in W^{1,2}[a,b], \tag{1.21}$$

  for some constant $C > 0$ independent of f.
- $C^1[a,b] \subset W^{1,2}[a,b]$ and is dense; this allows an alternative useful definition of $W^{1,2}[a,b]$ as the closure of $C^1[a,b]$ with respect to the above norm.

The main result for IBVP (1.1) in the case of Dirichlet, Neumann, and Mixed boundary conditions is based on the observation that the set of derivatives of the eigenfunctions, $X' = \{X_n'(x)\}$ is an orthogonal set in $L_k^2[a,b]$. This is used to determine the precise Sobolev setting for which energy computations can be carried out using a Parseval identity. Let $C_d^1[a,b]$ be the closed subspace of all $f \in C^1[a,b]$ that satisfy any Dirichlet boundary conditions present in the BVP (1.3). The appropriate setting is the space $W_d^{1,2}[a,b]$, the closure of $C_d^1[a,b]$ with respect to the norm (1.20); $W_d^{1,2}[a,b]$ is a Hilbert space with norm and inner product inherited from $W^{1,2}[a,b]$.

Theorem 1.2. *Let IBVP (1.1) have boundary conditions of type (D), (N) or (M). If $f \in W_d^{1,2}[a,b]$, then*

1. *the series (1.7) is uniformly convergent on $[a,b] \times [0,\infty)$, $u(x,t)$ is a weak solution of the wave equation with $u_t(\cdot,t) \in L_\rho^2[a,b]$ and $u(\cdot,t) \in W_d^{1,2}[a,b]$ for all $t > 0$;*
2. *the kinetic, potential, and total conserved energies are given by the respective formulas*

$$KE(t) = \frac{1}{2}\sum_{n=1}^{\infty} \mu_n\, \widehat{f_X}(n)^2 \sin^2\ \sqrt{\mu_n}t\, \|X_n\|_{2,\rho}^2 \tag{1.22}$$

$$PE(t) = \frac{1}{2}\sum_{n=1}^{\infty} \mu_n\, \widehat{f_X}(n)^2 \cos^2\ \sqrt{\mu_n}t\, \|X_n\|_{2,\rho}^2 \tag{1.23}$$

$$E = \frac{1}{2}\sum_{n=1}^{\infty} \mu_n\, \widehat{f_X}(n)^2 \|X_n\|_{2,\rho}^2 = \|f'\|_{2,k}^2; \tag{1.24}$$

3. *there is a uniformly almost periodic function $g(t)$ with mean value zero such that*

$$KE(t) = \frac{E}{2} - g(t),\ PE(t) = \frac{E}{2} + g(t).$$

Notice that the solution provided in the above theorem will satisfy any Dirichlet boundary condition present but may not satisfy a Neuman condition as the following example illustrates.

Example 1.3. *Let $a = 0$, $b = \pi/2$, $\theta_1 = 0$, $\theta_2 = \pi/2$ and take $k = \rho = 1$. For an initial condition take $f(x) = x$. Using the method of reflection, we extend this this problem to the interval $[0,\pi]$ by extending the initial data to be even about $x = \pi/2$. The initial data for the extended problem then has a singularity in the derivative at $x = \pi/2$ and further, this singularity propagates in the solution. Restricting the solution to the original interval, we see that the Neuman boundary condition at $\pi/2$ will not be satisfied. None the less, $u(x,t)$ is a weak solution of the wave equation built from the IBVP, has finite energy satisfying the conclusions of the above theorem.*

In the case where BVP (1.3) has a Robin boundary condition, the set $\{X_n'(x)\}$ is characteristically not orthogonal in $L_k^2[a,b]$. None the less, a result with similar conclusions as above can be obtained by imposing stronger smoothness on the initial data as follows. Let $W^{2,2}[a,b]$ be the Sobolev space of all functions $f \in L^2[a,b]$ with weak derivatives up to order two in $L^2[a,b]$. With inner product and norm given by

$$\begin{aligned}(f,g)_{W^{2,2}} &= (f,g)_{W^{1,2}} + (f'',g'')_\rho \\ \|f\|^2_{W^{2,2}} &= \|f\|^2_{2,\rho} + \|f'\|^2_{2,k} + \|f''\|^2_{2,\rho},\end{aligned} \tag{1.25}$$

$W^{2,2}[a,b]$ is a Hilbert space, the norm being equivalent to the usual one (with all weights equal to one) due to the conditions on the functions k and ρ. Then standard theory gives that $C^2[a,b]$ is dense in $W^{2,2}[a,b]$ and moreover, $W^{2,2}[a,b]$ is the closure of $C^2[a,b]$ with respect to the above norm. Let $C^2_{bc}[a,b]$ be the subspace of all $C[a,b]-$functions that satisfy the boundary conditions in BVP (1.3) and let $W^{2,2}_{bc}[a,b]$ be the closure of $C^2_{bc}[a,b]$ with respect to the $W^{2,2}-$norm. Then $W^{2,2}_{bc}[a,b]$ is a Hilbert space with inner product and norm inherited from $W^{2,2}[a,b]$. The following result valid for all types of boundary conditions, in particular Robin type boundary conditions.

Theorem 1.4. *Consider the IBVP (1.1) with any boundary conditions. If $f \in W^{2,2}_{bc}[a,b]$, then*

1. *the series (1.7) is uniformly convergent on $[a,b] \times [0,\infty)$, $u(x,t)$ is a weak solution of the wave equation with $u_t(\cdot,t) \in L^2_\rho[a,b]$ and $u(\cdot,t) \in W^{2,2}_{bc}[a,b]$ for all $t > 0$;*
2. *the kinetic and potential energies are given by (1.10) and (1.11) respectively and the total conseved energy is given by*

$$\begin{aligned}E &= \frac{1}{2}\sum_{n=1}^{\infty} \mu_n \widehat{f_X}(n)^2 \|X_n\|^2_{2,\rho} \\ &= \frac{1}{2}\|f'\|^2_{2,k} + \frac{1}{2}\left[k(b)\cot\theta_2 f^2(b) + k(a)\cot\theta_1 f^2(a)\right],\end{aligned}$$

   where the convention of Remark 1.1 is to be applied;
3. *energy equipartition in the form of (3) of Theorem 1.2 holds.*

The formulas for kinetic and potential energy in Theorems 1.2 and 1.4 are striking as they unify energy computations via Fourier analysis for all classical boundary conditions and seem to have been overlooked in the literature. The model example is a special case of Theorem 1.2 except in regards to energy equipartition. Periodicity of the kinetic and potential energy apparently occurs only for boundary conditions of the type considered in the theorem and when the coefficients $k(x)$ and $\rho(x)$ are constant. In contrast, note that sufficent smoothness is present in Theorem 1.4 to guarentee the solution satisfies the boundary conditions. Section 2 contains results of independent interest on eigenfunction expansions and Parseval type results for the spaces $W^{1,2}_d[a,b]$ and $W^{2,2}_{bc}[a,b]$ and supplies the proof of Theorem 1.2. The ideas in the Parseval theorem for $W^{2,2}_{bc}[a,b]$ (see Theorem 2.8) motivate a natural definition for fractional Sobolev spaces in the context of boundary value problems that lead to generalizing Theorem 1.2 to *all boundary conditions* considered in this paper as well as relaxing the smoothness hypothesis in Theorem 1.4. Section 3 is devoted to these unifying generalizations and provides extension of the results to the IBVP (1.1) where the initial conditions are replaced by

$$\text{IC: } \begin{cases} u(x,0) = 0 \\ u_t(x,0) = g(x) \end{cases}. \tag{1.26}$$

Remark 1.2. *Several additional comments are in order pertinent to the content of this paper.*

1. *Energy equipartition ideas seem to appear first in [4, 8] in the realm of abstract wave equations of form $u_{tt} = Au$, where A is a positive self adjoint operator on a Hilbert space with continuous spectrum. In the setting of this paper, the spectrum is discrete and the appearance of almost periodic functions is natural. The notion of uniform almost periodic function was introduced by Bohr (see [3] or [2]). The only fact from the theory we use is Bohr's theorem that a uniform limit of a sequence of uniformly almost periodic functions results in a uniformly almost periodic function.*
2. *This paper is devoted to a general form of the wave equation in one spatial dimension. Many of the results can be extended to higher dimensions, a topic of a forthcoming article.*
3. *Throughout this paper the symbol C will be used to denote a constant independent of parameters and functions involved in various estimates and it's value typically changes with each occurance.*

2. Parseval Theorems in Sobolev Spaces

2.1. Boundary Conditions of Dirichlet, Neumann, and Mixed Type

The following lemma is a key observation.

Lemma 2.1. *Consider the boundary value problem (1.3) with boundary conditions of type (D), (N), or (M). The the set of derivatives of the eigenfunctions $X' = \{X_n'\}_{n=1}^{\infty}$ forms an orthogonal set in $L^2_k[a,b]$. Moreover,*

$$\|X_n'\|_{2,k}^2 = \mu_n \|X_n\|_{2,\rho}^2. \tag{2.1}$$

Proof. This is an exercise in integration by parts providing a formula that is useful later:

$$\begin{aligned}\int_a^b X_n'(x)X_m'(x)k(x)dx &= X_n(b)X_m'(b)k(b) - X_n(a)X_m'(a)k(a) \\ &\quad - \int_a^b X_n(x)\left(k(x)X_m'(x)\right)' dx \\ &= X_n(b)X_m'(b)k(b) - X_n(a)X_m'(a)k(a) \\ &\quad + \mu_m \int_a^b X_n(x)X_m(x)\rho(x)dx.\end{aligned}$$

The result follows applying the boundary conditions and the orthogonality of the eigenfunctions in $L^2_\rho[a,b]$. □

Given $f \in W^{1,2}[a,b]$ with Fourier expansion

$$f \sim \sum_{n\in\Lambda} \widehat{f_X}(n) X_n(x),$$

a basic problem is to relate the formally derived series

$$\sum_{n\in\Lambda} \widehat{f_X}(n) X_n'(x)$$

to the Fourier expansion of f',

$$f' \sim \sum_{n=1}^{\infty} \widehat{f'}_{X'}(n) X_n'(x).$$

To address this, let $f \in C^1[a,b]$ and apply integration by parts: for $n \geq 1$,

$$\begin{aligned}\widehat{f'}_{X'}(n) &= \frac{1}{\|X_n'\|_{2,k}^2} \int_a^b f'(x) X_n'(x) k(x) dx \\ &= \frac{1}{\|X_n'\|_{2,k}^2} \left(f(x) X_n'(x) k(x) \Big|_a^b + \frac{\mu_n}{\|X_n'\|_{2,k}^2} \int_a^b f(x) X_n(x) \rho(x)\, dx. \end{aligned}$$

Rewritten this becomes

$$\widehat{f'}_{X'}(n) = \mathrm{BT}(n) + \widehat{f}_X(n), \tag{2.2}$$

where

$$\mathrm{BT}(n) = \frac{1}{\|X_n'\|_{2,k}^2} \Big(k(b) f(b) X_n'(b) - k(a) f(a) X_n'(a) \Big). \tag{2.3}$$

By density argument and the fact that $f \in C[a,b]$, (2.2) holds for $f \in W^{1,2}[a,b]$. A useful formula results if $\mathrm{BT}(n) = 0$ for all n and this can be arranged by forcing the function $f(x)$ to satisfy any Dirichlet condition present in the boundary conditions (if any). In fact, this condition is also necessary.

Lemma 2.2. *Let $f \in W^{1,2}[a,b]$. Then $\widehat{f'}_{X'}(n) = \widehat{f}_X(n)$, $n = 1, 2, \ldots$ if and only if:*

(D-BC) f satisfies any Dirichlet boundary condition present in (1.3).

Proof. Sufficiency of the condition being clear, we turn to necessity of the condition. If the boundary conditions are Neumann, the result is trivial-no other conditions on f are needed. In a mixed boundary condition problem such as $X(a) = 0$, $X'(b) = 0$, then for $\mathrm{BT}(n) = 0$ we must have $f(a)X_n'(a) = 0$, for all n. It follows that $f(a) = 0$ as $X_n'(a)$ cannot be zero. The argument for the other mixed case is similar. In the case of Dirichlet boundary conditions, we apply a consequence of the Sturm oscillation theorem [6] that the n^{th}−eigenfunction $X_n(x)$ has precisely $n-1$ roots in the interval (a,b). For $n = 1$, this implies that X_1' has opposite sign at both endpoints (notice that X_1' cannot be zero at either endpoint). Hence from $\mathrm{BT}(1) = 0$, $f(b) = c\, f(a)$ for some $c < 0$. Applying the same reasoning to the second eigenfunction leads to the conclusion that $f(b) = d\, f(a)$ for some $d > 0$. Taken together, it follows that $f(a) = f(b) = 0$. □

The above lemma essentially motivates the definition of the Hilbert space $W_d^{1,2}[a,b]$ in the introduction. The following in a restatement of Lemma 2.2 in useful form.

Proposition 2.3. *Let $f \in W^{1,2}[a,b]$. Then the following are equivalent:*

1. $f \in W_d^{1,2}[a,b]$;
2. $\widehat{f'}_{X'}(n) = \widehat{f}_X(n)$, $n = 1, 2, \ldots$.

Remark 2.1. *A special case of item (2) above is the classical formula (1.17).*

Proposition 2.4. *Let BVP (1.3) have boundary conditions of type (D), (N), or (M). Then the collection of eigenfunctions $\{X_n\}_{n \in \Lambda}$ is a complete orthogonal set in $W_d^{1,2}[a,b]$.*

Proof. Orthogonality follows from Lemma 2.1. Suppose $f \in W_d^{1,2}[a,b]$ with $(f, X_n)_{W^{1,2}} = 0$ for all n. Then using (2.1),

$$\begin{aligned}(f, X_n)_{W^{1,2}} &= \int_a^b f(x)X_n(x)\rho(x)dx + \int_a^b f'(x)X_n'(x)k(x)dx \\ &= \|X_n\|_{2,\rho}^2 \widehat{f_X}(n) + \|X_n'\|_{2,k}^2 \widehat{f'}_{X'}(n) \\ &= \|X_n\|_{2,\rho}^2 (1+\mu_n)\widehat{f_X}(n). \end{aligned} \tag{2.4}$$

It follows that $\widehat{f_X}(n) = 0$ for all $n \in \Lambda$ and hence $f = 0$ as the orthogonal set $\{X_n\}$ is complete in $L_\rho^2[a,b]$. □

As a consequence, Parseval's equality holds: if $f \in W_d^{1,2}[a,b]$, then

$$\|f\|_{W^{1,2}}^2 = \sum_{n\in\Lambda} \widehat{f}(n)^2 \|X_n\|_{W^{1,2}}^2, \tag{2.5}$$

where

$$\widehat{f}(n) = \frac{(f, X_n)_{W^{1,2}}}{\|X_n\|_{W^{1,2}}^2},\ n \in \Lambda.$$

From (2.4) and $\|X_n\|_{W^{1,2}}^2 = (1+\mu_n)\|X_n\|_{2,\rho}^2$ it follows that $\widehat{f}(n) = \widehat{f_X}(n)$, for all n. This leads to the following reformulation of the above Parseval relation in a form useful for energy computations.

Theorem 2.5. *Let BVP (1.3) have boundary conditions of type (D), (N), or (M). If $f \in W_d^{1,2}[a,b]$ then*

$$\sum_{n\in\Lambda} \mu_n \widehat{f_X}(n)^2 \|X_n\|_{2,\rho}^2 < \infty, \tag{2.6}$$

$$\|f\|_{W^{1,2}}^2 = \sum_{n\in\Lambda} (1+\mu_n)\widehat{f_X}(n)^2 \|X_n\|_{2,\rho}^2, \tag{2.7}$$

and in particular,

$$\|f'\|_{2,k}^2 = \sum_{n=1}^{\infty} \mu_n \widehat{f_X}(n)^2 \|X_n\|_{2,\rho}^2 = \sum_{n=1}^{\infty} \widehat{f'}_{X'}(n)^2 \|X_n'\|_{2,k}^2.$$

Conversely, if $\{c_n\}_{n\in\Lambda} \subset \mathbb{R}$ is a sequence satisfing

$$\sum_{n=1}^{\infty} n^2 c_n^2 \|X_n\|_{2,\rho}^2 < \infty, \tag{2.8}$$

then there is $f \in W_d^{1,2}[a,b]$ such that $c_n = \widehat{f_X}(n)$, $n \in \Lambda$.

Proof. For the first statement, $f' \in L_k^2[a,b]$ and Bessel's inequality give

$$\sum_{n=1}^{\infty} \widehat{f'}_{X'}(n)^2 \|X_n'\|_{2,k}^2 < \infty.$$

The latter series is the same as (2.6) by Proposition 2.3 and (2.1). The Parseval identity (2.7) then follows by the preceeding discussion. For the converse, the hypothesis implies $\sum_{n\in\Lambda} c_n^2 \|X_n\|_{2,\rho}^2 < \infty$

and by Parseval's theorem for eigenfunction expansions, $c_n = \widehat{f_X}(n)$, $n \in \Lambda$ for some $f \in L^2_\rho[a,b]$. It remains to prove that the weak derivative $f' \in L^2_k[a,b]$. Again by [6] , $\mu_n \sim O(n^2)$ $(n \to \infty)$, and hence (2.8) implies $\sum_n \mu_n \widehat{f_X}(n)^2 \|X_n\|^2_{2,\rho} < \infty$. Let

$$F_N(x) = \sum_{n \le N} \widehat{f_X}(n) X_n(x)$$

be the partial sums of the eigenfunction expansion of f. Then $\|F_N - f\|_{2,\rho} \to 0$ as $N \to \infty$ since $f \in L^2_\rho[a,b]$. Since $F'_N(x) = \sum_{n \le N} \widehat{f_X}(n) X'_n(x)$, Lemma 2.1 and standard computation gives for $N > M$,

$$\|F'_N - F'_M\|^2_{2,k} = \sum_{n=M+1}^{N} \widehat{f_X}(n)^2 \|X'_n\|^2_{2,k} = \sum_{n=M+1}^{N} \mu_n \widehat{f_X}(n)^2 \|X_n\|^2_{2,\rho} \to 0,$$

as $M, N \to \infty$. Hence, there is $g \in L^2_k[a,b]$ such that $\|F'_N - g\|_{2,k} \to 0$ as $N \to \infty$. The claim is $g = f'$, the weak derivative of f. Indeed, if $\phi \in C^\infty_c(a,b)$, then

$$\begin{aligned}\int_a^b g(x)\phi(x)dx &= \lim_{N\to\infty} \int_a^b F'_N(x)\phi(x)dx = \lim_{N\to\infty} \sum_{n \le N} \widehat{f_X}(n) \int_a^b X'_n(x)\phi(x)dx \\ &= -\lim_{N\to\infty} \sum_{n \le N} \widehat{f_X}(n) \int_a^b X_n(x)\phi'(x)dx = -\int_a^b f(x)\phi'(x)dx.\end{aligned}$$

It now follows that $\|F_N - f\|_{W^{1,2}} \to 0$ as $N \to \infty$ and the result follows since $F_N \in C^2_{bc}[a,b] \subset C^1_d[a,b]$. □

The focus of Proposition 2.3 and Theorem 2.5 is the relation between differentiating an eigenfunction expansion term by term and Parseval's equality for $W^{1,2}_d[a,b]$, ideas that underly our study of IBVP (1.1) for boundary conditions of Dirichlet, Neumann, or mixed type.

All the needed machinery is in place to supply the following.

Proof of Theorem 1.2. From $f \in W^{1,2}_d[a,b]$ follows for all $t \ge 0$,

$$\begin{aligned}&\sum_{n\in\Lambda} \widehat{f_X}(n)^2 \cos^2 \sqrt{\mu_n} t \, \|X_n\|^2_{2,\rho} < \infty \text{ and} \\ &\sum_{n=1}^{\infty} \widehat{f_X}(n)^2 \cos^2 \sqrt{\mu_n} t \, \|X'_n\|^2_{2,k} = \sum_{n=1}^{\infty} \mu_n \widehat{f_X}(n)^2 \cos^2 \sqrt{\mu_n} t \, \|X_n\|^2_{2,\rho} < \infty.\end{aligned}$$

Consequently the series

$$u(x,t) = \sum_{n\in\Lambda} \widehat{f_X}(n) \cos \sqrt{\mu_n} t \, X_n(x) \tag{2.9}$$

converges in $L^2_\rho[a,b]$ for all $t \ge 0$ and by Theorem 2.5 is the Fourier eigenfunction expansion of $x \to u(x,t)$ and this function is in $W^{1,2}_d[a,b]$. Further, the series of term by term derivatives converges in $L^2_k[a,b]$ and defines the weak derivative $u_x(x,t)$. For $N \in \mathbb{N}$, let

$$u_N(x,t) = \sum_{n \le N} \widehat{f_X}(n) \cos \sqrt{\mu_n} t \, X_n(x).$$

By Sobolev's inequality, for a constant $C > 0$,

$$\|u_N(\cdot,t) - u(\cdot,t)\|_\infty^2 \le C\, \|u_N(\cdot,t) - u(\cdot,t)\|_{W^{1,2}}^2, \tag{2.10}$$

and using Corollary 2.5 and simple estimate we have

$$\sup_{t\ge 0} \|u_N(\cdot,t) - u(\cdot,t)\|_\infty^2 \le C \sum_{n=N+1}^{\infty} (1+\mu_n)\widehat{f_X}(n)^2 \|X_n\|_{2,\rho}^2.$$

Since the right hand side tends to zero as $N \to \infty$, we have proved (1) in Theorem 1.2. The series of term by term derivative of (2.9) with respect to t converges in L^2_ρ−norm to the weak derivative $u_t(x,t)$ and the formula for the kinetic energy (1.10) follows from Theorem 1.1. The potential energy formula (1.11) is now immediate from Theorem 2.5. The proof of (2) in Theorem 1.2 is complete. For (3) in the theorem, using the half-angle formulas for sine and cosine in the formulas for kinetic and potential energy give

$$\mathrm{KE}(t) = E/2 - g(t) \text{ and } \mathrm{PE}(t) = E/2 + g(t),$$

where

$$g(t) = \frac{1}{4}\sum_{n=1}^{\infty} \mu_n \widehat{f_X}(n)^2 \|X_n\|_{2,\rho}^2 \cos 2\sqrt{\mu_n} t.$$

The latter series is uniformly convergent on $[0,\infty)$ to a uniformly almost periodic function. Integrating term by term,

$$\begin{aligned}\left|\frac{1}{T}\int_0^T g(t)dt\right| &= \frac{1}{2T}\left|\sum_{n=1}^{\infty} \mu_n^{1/2} \widehat{f_X}(n)^2 \|X_n\|_{2,\rho}^2 \sin 2\sqrt{\mu_n} T\right| \\ &\le \frac{1}{2T}\sum_{n=1}^{\infty} \mu_n^{1/2} \widehat{f_X}(n)^2 \|X_n\|_{2,\rho}^2\end{aligned}$$

and it follows that g has mean value zero. This concludes the proof of Theorem 1.2. □

2.2. Boundary Conditions in General

As mentioned in the introduction, in the case where the boundary conditions in BVP (1.3) are of Robin type, the set $\{X_n'(x)\}$ is characteristically not orthogonal in $L^2_k[a,b]$. This can be seen via the formula obtained in the proof of Lemma 2.1:

$$\int_a^b X_n'(x)X_m'(x)k(x)dx = X_n(b)X_m'(b)k(b) - X_n(a)X_m'(a)k(a).$$

In the case where there is a Robin boundary condition at one end point, say $x = b$, and a Dirichlet or Neumann condition at $x = a$, the result follows as neither X_l or X_l' can equal zero at $x = b$ for all l. When there is a Robin condition at both ends, applying the boundary conditions the above formula takes the form

$$\int_a^b X_n'(x)X_m'(x)k(x)dx = -\left[k(b)\cot\theta_2 X_n(b)X_m(b) + k(a)\cot\theta_1 X_n(a)X_m(a)\right].$$

If n and m have the same parity, then arguments used earlier based on the zeros of the eigenfunctions forces both terms in the bracket to have the same sign and hence the right hand side cannot equal zero.

Independent of this fact, stronger smoothness assumptions and boundary control lead to formulas useful in computing potential energy. Within this subsection analogs of Lemma 2.2, Proposition 2.3, and Theorem 2.5 will be obtained for $W^{2,2}[a,b]$.

Introducing the differential operator

$$L = -\frac{1}{\rho(x)}\frac{d}{dx}\left(k(x)\frac{d}{dx}\right),$$

then applying integration by parts twice with $f \in C^2[a,b]$ yields

$$\widehat{Lf}_X(n) = \mathrm{BT}(n) + \mu_n \widehat{f}_X(n), \tag{2.11}$$

where

$$\mathrm{BT}(n) = \frac{1}{\|X_n\|_{2,\rho}^2}\left\{k(b)\left[f(b)X_n'(b) - f'(b)X_n(b)\right] + k(a)\left[f'(a)X_n(a) - f(a)X_n'(a)\right]\right\}. \tag{2.12}$$

If $f \in W^{2,2}[a,b]$, then f and $f' \in C[a,b]$ and $Lf \in L^2[a,b]$. Hence by a density argument the above formula is valid on $W^{2,2}[a,b]$. We are interested in the useful case when $\mathrm{BT}(n) = 0$ for all n.

Lemma 2.6. *Consider BVP (1.3) with any of the aforementioned boundary conditions. Let $f \in W^{2,2}[a,b]$. Then $\widehat{Lf}_X(n) = \mu_n \widehat{f}_X(n)$, $n = 1, 2, \ldots$ if and only if f satisfies the boundary conditions in BVP (1.3).*

Proof. Sufficiency is easily shown applying the boundary conditions in (2.12). For the proof of necessity there are nine cases to be considered and they are all similar. To exempliy, the proof is provided in the case of a Robin boundary condition at both endpoints. In this case, applying the boundary conditions satisfied by the eigenfunctions and assuming $\mathrm{BT}(n) = 0$ gives

$$k(b)X_n(b)\left[g(b)\cot\theta_2 + g'(b)\right] + k(a)X_n(a)\left[g'(a) - g(a)\cot\theta_1\right] = 0.$$

The sign of each term in the above sum is determined by the parity of n and the same argument used in Lemma 2.2 yields the result. □

The analytically useful reformulation of the above lemma and analog of Proposition 2.3 is as follows.

Proposition 2.7. *Consider BVP (1.3) with any boundary conditions and let $f \in W^{2,2}[a,b]$. Then the following are equivalent.*

1. *$f \in W^{2,2}_{bc}[a,b]$;*
2. *$\widehat{Lf}_X(n) = \mu_n \widehat{f}_X(n)$, $n = 1, 2, \ldots$;*

The following theorem essentially provides a Parseval theorem for $W^{2,2}_{bc}[a,b]$ and can be used for energy computations with Robin boundary conditions. In the case of boundary conditions of type (D), (N), or (M) it is a weaker result than Theorem 2.5 due to stronger smoothness hypothesis.

Theorem 2.8. *Consider BVP (1.3) with any boundary conditions. If* $f \in W^{2,2}_{bc}[a,b]$*, then*

$$\sum_{n=1}^{\infty} \mu_n^2 \widehat{f_X}(n)^2 \|X_n\|^2_{2,\rho} < \infty \tag{2.13}$$

and

$$\|f'\|^2_{2,k} + k(b)\cot\theta_2 f^2(b) + k(a)\cot\theta_1 f^2(a) = \sum_{n=1}^{\infty} \mu_n \widehat{f_X}(n)^2 \|X_n\|^2_{2,\rho}. \tag{2.14}$$

Conversely, if $\{c_n\} \subset \mathbb{R}$ *such that*

$$\sum_{n=1}^{\infty} n^4 c_n^2 \|X_n\|^2_{2,\rho} < \infty,$$

then there is $f \in W^{2,2}_{bc}[a,b]$ *such that* $c_n = \widehat{f_X}(n)$*,* $n \in \Lambda$*.*

Proof. Since $Lf \in L^2_\rho[a,b]$, (2.13) follows from Parseval's theorem for eigenfunction expansion with respect to $\{X_n\}_{n\in\Lambda}$ and Proposition 2.7. If $f \in C^2_{bc}[a,b]$, integration by parts and the boundary conditions yeild the identity

$$\int_a^b f(x)Lf(x)\rho(x)dx = \|f'\|^2_{2,k} + k(b)\cot\theta_2 f^2(b) + k(a)\cot\theta_1 f^2(a).$$

This formula is valid for $f \in W^{2,2}_{bc}[a,b]$ by a density argument. Applying the dual form of Parseval's equality (1.6) to the left hand side then concludes the proof of (2.14). The converse is obtained in a fashion similar to that of Theorem 2.5 with the following additions. Using the notation in that proof, we have that $\|F_N - f\|_{2,\rho} \to 0$ as $N \to \infty$ and the sequence $\{LF_N\}_{n=1}^{\infty}$ converges in L^2_ρ–norm. Let $N > M + 1$ and consider the estimate

$$\|F'_N - F'_M\|_{2,k} \leq \sum_{n=M+1}^{N} |\widehat{f_X}(n)|\, \|X'_n\|_{2,k}.$$

In the case of boundary conditions of type (D), (N), or (M), $\|X'_n\|_{2,k} = \sqrt{\mu_n}\|X_n\|_{2,\rho}$ from Lemma 2.1. Assuming a boundary condition of type (R), using the identity in the proof of this Lemma gives the estimate $\|X'_n\|_{2,k} \leq \sqrt{\mu_n}\|X_n\|_{2,\rho}$. It follows for all boundary conditions that

$$\|F'_N - F'_M\|_{2,k} \leq \sum_{n=M+1}^{N} \sqrt{\mu_n}\, |\widehat{f_X}(n)|\, \|X_n\|_{2,\rho}.$$

Applying the Cauchy-Schwarz inequality we have

$$\|F'_N - F'_M\|_{2,k} \leq \left(\sum_{n=M+1}^{N} \frac{1}{\mu_n}\right)^{1/2} \left(\sum_{n=M+1}^{N} \mu_n^2 \widehat{f_X}(n)^2 \|X_n\|^2_{2,\rho}\right)^{1/2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{\mu_n} < \infty$, it follows that the sequence $\{F'_N\}$ is Cauchy in $L^2_k[a,b]$ and hence converges to a function, say $g \in L^2_k[a,b]$. Arguments used previously show that $g = f'$ and thus $f' \in L^2_k[a,b]$ with $\|F'_N - f'\|_{2,k} \to 0$ as $N \to \infty$. We have

$$F''_N = \frac{k'F'_N - \rho(LF_N)}{k},$$

and by the conditions on k and ρ,

$$\|F_N'' - F_M''\|_{2,\rho} \leq C\left(\|F_N' - F_M'\|_{2,k} + \|LF_N - LF_M\|_{2,\rho}\right),$$

for some constant C. Hence there is $h \in L^2_\rho[a,b]$ such that $\|F_N'' - h\|_{2,\rho} \to 0$ as $N \to \infty$. By a duality argument $h = f''$ and we can conclude that $\|F_N - f\|_{W^{2,2}} \to 0$ as $N \to \infty$. The proof is concluded with the observation that $F_N \in C^2_{bc}[a,b]$. □

The above result can be used to prove Theorem 1.4 in a fashion similar to the proof of Theorem 1.2.

3. Fractional Sobolev Spaces: Unification & Variations

Fractional Sobolev spaces associated with a particular Sturm-Liouville problem were introduced in [1]. Our treatment below is similar, applies to all Sturm-Liouville problems (1.3), is well motivated by the form of the energy calculations of interest in this paper, and serves to unify Theorem 1.2 and Theorem 1.4.

The conclusion of Theorem 2.8 (2.14) suggests a simplier description of $W^{2,2}_{bc}[a,b]$ that lends itself to defining fractional powers of the operator L and associated fractional Sobolev spaces. Indeed, we see that $W^{2,2}_{bc}[a,b]$ is the set of all $f \in L^2_\rho[a,b]$ such that (2.13) holds. The norm $\|\cdot\|_{(2,2)}$ defined by

$$\|f\|^2_{(2,2)} = \|f\|^2_{2,\rho} + \|Lf\|^2_{2,\rho},$$

is equivalent to the norm on $W^{2,2}[a,b]$ given by (1.25). Furthermore, the action of L on $W^{2,2}_{bc}[a,b]$ can be understood through its spectral representation:

$$Lf(x) = \sum_{n=1}^{\infty} \mu_n \widehat{f_X}(n) X_n(x),$$

convergence in L^2_ρ−norm. This suggests formally defining for $s \geq 0$,

$$L^{s/2} f(x) = \sum_{n=1}^{\infty} \mu_n^{s/2} \widehat{f_X}(n) X_n(x)$$

and defining

$$\begin{aligned}\mathcal{H}^s[a,b] &= \{f \in L^2[a,b] \,|\, L^{s/2} f \in L^2_\rho[a,b]\} \\ &= \{f \in L^2[a,b] \,|\, \sum_{n=1}^{\infty} \mu_n^s \widehat{f_X}(n)^2 \|X_n\|^2_{2,\rho} < \infty\}.\end{aligned}$$

The latter equality holds by Parseval's Theorem 1.1. Then with norm $\|\cdot\|_{(2,s)}$ defined by

$$\begin{aligned}\|f\|^2_{(s,2)} &= \|f\|^2_{2,\rho} + \|L^{s/2} f\|^2_{2,\rho} \\ &= \sum_{n\in\Lambda} (1 + \mu_n^s) \widehat{f_X}(n)^2 \|X_n\|^2_{2,\rho}.\end{aligned}$$

and obvious choice for inner product it is easily shown that $\mathcal{H}^s[a,b]$ is a Hilbert space, $\mathcal{H}^2[a,b] = W^{2,2}_{bc}[a,b]$, and $\mathcal{H}^0[a,b]$ is identified with $L^2_\rho[a,b]$. Also note that $C^2_{bc}[a,b]$ is a dense subset of $\mathcal{H}^s[a,b]$ and

$$\mathcal{H}^s[a,b] \supset \mathcal{H}^t[a,b] \text{ if } s < t. \tag{3.1}$$

Here is an analog of the Sobolev embedding theorem.

Lemma 3.1. *Let $s > 1/2$. Then $\mathcal{H}^s[a,b] \subset C[a,b]$, $f \in \mathcal{H}^s[a,b]$ satisfies condition (D-BC) of Lemma 2.2, and*

$$\|f\|_\infty \le C\,\|f\|_{(s,2)}, \tag{3.2}$$

for some constant $C > 0$.

Proof. Let $f \in \mathcal{H}^s[a,b]$. It will be shown that the Fourier eigenfunction expansion of f is uniformly convergent. Let $Y_n = X_n/\|X_n\|_{2,\rho}$, then $Y = \{Y_n\}_{n\in\Lambda}$ is a complete set of orthonormal eigenfunctions in $L^2_\rho[a,b]$. It follows that $\widehat{f_Y}(n) = \|X_n\|_{2,\rho}\widehat{f_X}(n)$. The advantage of passing to the orthonormal eigenfunctions is that they are uniformly bounded, i.e., $|Y_n(x)| \le C$ (this follows from explicit asymptotics of Y_n, see [6] or [7]). Let $N > M$, then

$$\begin{aligned}\left|\sum_{n=M+1}^{N} \widehat{f_X}(n)X_n(x)\right| &= \left|\sum_{n=M+1}^{N} \widehat{f_Y}(n)Y_n(x)\right| \le C \sum_{n=M+1}^{N} |\widehat{f_Y}(n)| \\ &= C \sum_{n=M+1}^{N} |\widehat{f_Y}(n)|\mu_n^{s/2}\mu_n^{-s/2} \le C\left(\sum_{n=M+1}^{N} \mu_n^s \widehat{f_Y}(n)^2\right)^{1/2}\left(\sum_{n=M+1}^{N} \mu_n^{-s}\right)^{1/2} \\ &\le C\left(\sum_{n=M+1}^{N} \mu_n^s \widehat{f_X}(n)^2\|X_n\|^2_{2,\rho}\right)^{1/2}\end{aligned}$$

where the Cauchy-Schwarz inequality was applied and $\sum_{n=1}^\infty \mu_n^{-s} < \infty$ provided $s > 1/2$. This inequality shows that the sequence of partial sums is uniformly Cauchy on $[a,b]$ and completes the proof that $f \in C[a,b]$. That f satisfies condition (D-BC) follows as the sequence of partial sums satisfy the boundary conditions. If the boundary conditions are not of type (N), we set $M = 0$ in the above estimate to deduce (3.2). If the boundary conditions are of type (N), there is an additional term corresponding to the zero eigenvalue; this term is easily bounded above by a constant times the L^2_ρ−norm of f. This completes the proof. □

Examples can be given of functions $f \in \mathcal{H}^s[a,b]$ for all $s < 1/2$ and which are discontinuous on $[a,b]$, see [1]. Strengthening the assumption on s implies greater smoothness on f as follows.

Lemma 3.2. *Let $s > 3/2$. If $f \in \mathcal{H}^s[a,b]$, then $f' \in C^1[a,b]$, f satifies the boundary conditions, and the following estimate holds:*

$$\|f'\|_\infty \le C\|L^{s/2}f\|_{2,\rho},$$

for some constant $C > 0$.

Proof. The proof is similar to that of the previous lemma where the uniform boundedness of the orthonormal set of eigenfunctions is replaced by the estimate [7] $|Y_n'(x)| \le C\mu_n^{1/2}$. Details are left to the reader. □

Proposition 3.3. *Let $s \geq 1$. If $f \in \mathcal{H}^s[a,b]$, then the weak derivative f' satisfies $f' \in L^2_k[a,b]$, and*

$$\|f'\|^2_{2,k} + k(b)\cot\theta_2 f^2(b) + k(a)\cot\theta_1 f^2(a) = \|L^{1/2} f\|^2_{2,\rho}. \tag{3.3}$$

Here we follow the convention that if a Dirichlet condition holds at one of the endpoints, the corresponding term on the left hand side is dropped (Remark 1.1).

Proof. Let $f \in \mathcal{H}^s[a,b]$. Because of (3.1), $f \in \mathcal{H}^1[a,b]$ and

$$\sum_{n=1}^{\infty} \mu_n \widehat{f_X}(n)^2 \|X_n\|^2_{2,\rho} = \|L^{1/2} f\|^2_{2,\rho} < \infty.$$

Let $F_N(x)$ be the N^{th}−partial sum of the Fourier eigenfunction expansion of f as introduced earlier. Then for $N > M$, $F_N - F_M \in C^2_{bc}[a,b]$ and from (2.14) we obtain

$$\|F'_N - F'_M\|^2_{2,k} + k(b)\cot\theta_2 \left(F_N(b) - F_M(b)\right)^2(b) + k(a)\cot\theta_1 \left(F_N(a) - F_M(a)\right)^2(a) = \sum_{n=M+1}^{N} \mu_n \widehat{f_X}(n)^2 \|X_n\|^2_{2,\rho}.$$

The right hand side tends to zero as $N, M \to \infty$ and by the proof of Lemma 3.1, the terms at $x = a$ and $x = b$ also tend to zero. Thus $\{F'_n\}_{n=1}^{\infty}$ is Cauchy in L^2_k−norm and hence converges; using previous arguments the limit is the weak derivative f'. Substituting F_N in for f in (2.14) and letting N tend to infinity demonstrates (3.3). □

Not only is equality (3.3) key to potential energy estimates, it also provides a unifying link to the space $W^{1,2}_d[a,b]$ as the following immediate corollary provides.

Corollary 3.4. *Let BVP (1.3) have boundary conditions of type (D), (N), or (M). Then $\mathcal{H}^1[a,b] = W^{1,2}_d[a,b]$ and the spaces have equivalent norms.*

The following theorem unifies Theorem 1.2 and Theorem 1.4 providing generalization of the former to all boundary conditions and weaking the smoothness hypothesis for the latter.

Theorem 3.5. *Consider IBVP (1.1) with any of the types of boundary conditions. If $f \in \mathcal{H}^s[a,b]$ for some $s \geq 1$, then*

1. *the series (1.7) is uniformly convergent on $[a,b] \times [0,\infty)$, $u(x,t)$ is a weak solution of the wave equation with $u_t(\cdot,t) \in L^2_\rho[a,b]$ and $u(\cdot,t) \in \mathcal{H}^s[a,b]$ for all $t > 0$;*
2. *the kinetic and potential energies are given by (1.10) and (1.11) respectively and the total conseved energy is given by*

$$E = \frac{1}{2}\sum_{n=1}^{\infty} \mu_n \widehat{f_X}(n)^2 \|X_n\|^2_{2,\rho} = \frac{1}{2}\|L^{1/2} f\|^2_{2,\rho};$$

3. *energy equipartition in the form of (3) of Theorem 1.2 holds.*

Proof. The proof is similar to that of Theorem 1.2 and somewhat simplier. Given $f \in \mathcal{H}^s[a,b]$ for some $s \geq 1$, all we must do is check that $u(\cdot,t) \in \mathcal{H}^s[a,b]$ for all $t \geq 0$. From the proof of Theorem 1.2, it is clear that $u(\cdot,t) \in L^2_\rho[a,b]$ and that $\widehat{u(\cdot,t)}_X(n) = \widehat{f}_X(n)\cos\sqrt{\mu_n}t$. Thus, $u(\cdot,t) \in \mathcal{H}^s[a,b]$ follows from the convergence of

$$\sum_{n=1}^{\infty} \mu_n^s \widehat{f}_X(n)^2 \|X_n\|_{2,\rho}^2.$$

Since $s \geq 1$, the potential energy can be computed using (3.3) with f replaced by $u(x,t)$:

$$\mathrm{PE}(t) = \frac{1}{2}\|L^{1/2}u(\cdot,t)\|_{2,\rho}^2 = \sum_{n=1}^{\infty} \mu_n \widehat{f}_X(n)^2 \cos^2\sqrt{\mu_n}t\,\|X_n\|_{2,\rho}^2.$$

The rest of the proof is carried out just as in the proof of Theorem 1.2. □

Remark 3.1. *If $s > 3/2$ in the preceeding theorem, then Lemma 3.2 applies and $u(\cdot,t) \in C^1[a,b]$ and it follows that $u(x,t)$ satisfies the boundary conditions. This generalizes Theorem 1.4.*

The final result of this paper provides the analog of the above theorem for IBVP (1.1) where the initial conditions are replaced by (1.26). The Fourier analytic form for the solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{\widehat{f}_X(n)}{\sqrt{\mu_n}} \sin\sqrt{\mu_n}t\, X_n(x). \tag{3.4}$$

In the case of Neumann boundary conditions, the assumption $\widehat{g}_X(0) = 0$ is additionally imposed. The proof of the following result is similar to that of the previous theorem and is left for the reader.

Theorem 3.6. *Consider IBVP (1.1) with initial conditions replaced by (1.26) where $g \in \mathcal{H}^s[a,b]$ for some $s \geq 0$. Then the following hold*

1. *the series (3.4) is uniformly convergent on $[a,b] \times [0,\infty)$, $u(x,t)$ is a weak solution of the wave equation with $u_t(\cdot,t) \in L^2_\rho[a,b]$ and $u(\cdot,t) \in \mathcal{H}^{s+1}[a,b]$ for all $t > 0$;*
2. *the kinetic, potential, and total conserved energies are given by*

$$KE(t) = \frac{1}{2}\sum_{n=1}^{\infty} \widehat{g}_X(n)^2 \cos^2\sqrt{\mu_n}t\,\|X_n\|_{2,\rho}^2$$

$$PE(t) = \frac{1}{2}\sum_{n=1}^{\infty} \widehat{g}_X(n)^2 \sin^2\sqrt{\mu_n}t\,\|X_n\|_{2,\rho}^2 = \frac{1}{2}\|L^{1/2}u(\cdot,t)\|_{2,\rho}^2$$

$$E = \frac{1}{2}\sum_{n=1}^{\infty} \widehat{g}_X(n)^2 \|X_n\|_{2,\rho}^2 = \frac{1}{2}\|g\|_{2,\rho}^2$$

3. *energy equipartition in the form of (3) of Theorem 1.2 holds.*

References

1. A.B. Belkaid, *Fractional power function spaces associated to regular Sturm-Liouville problems*, Elect. Jour. Diff. Eqs. **49** (2005), p. 1-12.

2. A.S. Bescovitch, *Almost Periodic Functions,* Dover Publ.,1959.
3. H. Bohr, *Almost Periodic Functions,* Chelsea Publ., 1951.
4. A.R. Brodsky, *On the asymptotic behaviour of solutions of wave equations*, Proc. Amer. Math. Soc. **18** (1967), p. 207-208.
5. W. O. Bray, *A Journey into Partial Differential Equations,* Jones & Bartlett Learning, 2012.
6. G. Birkhoff, G. Carlo-Rota, *Ordinary Differential Equations,* 4th edition, J. Wiley, 1989.
7. C.T. Fulton, S.A. Pruess, *Eigenvalue and eigenfunction asypmtotics for regular Sturm-Liouville problems*, Jour. Math. Anal. Appl. **188** (1994, p. 297-340.
8. J.A. Goldstein, *An asymptotic property of solutions of wave equations*, Proc. Amer. Math. Soc. **23** (1968), p. 359-363.
9. G. Leoni, *A First Course in Sobolev Spaces,* Graduate Studies in Mathematics, Vol 105, Amer. Math. Soc., 2009.
10. R.L. Wheeden, A. Zygmund, *Measure and Integral, An Introduction to Real Analysis,* 2ed., CRC Press, 2015.