On mathematical models with unknown nonlinear convection coefficients in one-phase heat transform processes

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Abstract

In this work, one-phase models for restoration of unknown temperature-dependent convection coefficients are considered by using the final observation of the temperature distribution and the phase boundary position. The proposed approach allows one to obtain sufficient conditions of unique identification of such coefficients in a class of smooth functions. Sets of admissible solutions preserving the uniqueness property are indicated. The considered mathematical models allow one to take into account the dependence of thermophysical characteristics upon the temperature.

Keywords: One-phase models, inverse restoration problems, quasilinear parabolic equations, Stefan problems, uniqueness property.

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1. Introduction

In a thermophysical interpretation, the one-phase models considered in the present work consist of finding the temperature field, phase transition boundary (e.g. the melting front), and the temperature-dependent convection coefficient under the assumption that the temperature distribution and the phase boundary position are given at a final time. As it is known, the determination of any causal characteristic by some measured effect characteristics in the corresponding physical process (in our case this is a convection coefficient) leads to inverse restoration problems. Just like most of the inverse tasks in mathematical physics, the inverse restoration problems (or the so-called identification problems) are ill-posed in the Tikhonov sense [1]. Their solution need not be unique and stable, i.e., continuous depending on the input data. This is a result of the violation of the cause-effect relations in their statements. Unlike inverse restoration problems for parabolic equations in domains with fixed boundaries, such problems for phase transform models are unsufficiently studied. Their study is more difficult because of the presence of the unknown moving phase boundary. Nevertheless, theoretical justification of this class of inverse restoration problems is motivated by the modern needs of technologies both in heat processes (e.g., power engineering, metallurgy, and astronautics) and in hydrology, exploitation of oil-gas fields, etc. Various statements of such problems with phase transforms, depending on the unknown causal characteristic and the type of additional information, are presented, for example, in [2–13].

The corresponding mathematical formulations are inverse Stefan problems for parabolic equations in domains with free boundaries with material or energy balance conditions imposed on them. This paper continues our investigation of such inverse problems begun in [14–17]. The present mathematical statements are connected with restoration of the unknown convection coefficient and consist in determining the unknown coefficient multiplying the lowest order derivative in a quasilinear parabolic equation in a one-phase domain whose external boundary is a phase front with an unknown time dependence. Additional information is given in the form of final overdetermination.

Our statements for the quasilinear parabolic equation allow one to take into account the dependence of thermophysical characteristics upon the temperature — such models arise, for example, in the modeling of the high temperature processes. In order to overcome instability of solutions in the present inverse restoration problems, the principles of constructing stable approximate solutions of ill-posed inverse Stefan problems from [14] are applicable. In the present paper, our main attention is given to the other difficulty connected with ill-posed inverse restoration problems. Namely, our aim is to obtain sufficient conditions of unique identification of the nonlinear convection coefficients.

To this end, in Section 2 we justify the mathematical statements of the corresponding inverse Stefan problems choosing function spaces for the input data and the solution of the restoration problems. This choice relies on the research of classical solvability of the corresponding direct Stefan problems. This is important for ill-posed inverse problems — if there is no coordination between the given input data, the exact solution of the inverse problem does not exist. In order to prove sufficient conditions of unique identification in a class of smooth functions, we use the duality principle by analogy with [18], where it was applied to a parabolic equation with an unknown coefficient multiplying the lowest order derivative in a domain with fixed boundaries. To this end, the "straightening phase boundaries" substitution is carried out, which transforms the phase domain into a rectangular domain of fixed width. In Sections 3 and 4 the proposed approach allows one to establish uniqueness theorems for the corresponding statements of the inverse restoration problems. In Section 5 sets of admissible solutions preserving the uniqueness property are indicated. It is shown that this property may be lost if the desired nonlinear convection coefficient depends not only on the temperature and the spatial variable but also on the time. Finally, a short conclusion in Section 6 summarizes the results obtained in this work.

The following remarks must be added.

In our analysis we use standard definitions for the function spaces from [19]. In particular, the following definitions are used.

$\overset{0}{C}[0,l]$ is the space of fuctions $u(x)$ continuous on the interval $[0,l]$ with $u|_{x=0} = 0$, $u|_{x=l} = 0$.

$H^{2+\lambda}[0,l]$ is the space of fuctions $u(x)$ continuous on the interval $[0,l]$ together with their derivatives u_{xx} which satisfy the Hölder condition with the exponent λ.

$H^{1+\lambda/2}[0,T]$ is the space of fuctions $u(t)$ continuous on the interval $[0,T]$ together with their derivatives u_t which satisfy the Hölder condition with the exponent $\lambda/2$.

$H^{\lambda,\lambda/2}(\overline{Q})$ is the space of fuctions $u(x,t)$ continuous on the closed set $\overline{Q} = \{0 \le x \le l, 0 \le t \le T\}$ which satisfy the Hölder conditions in x and t with the corresponding exponents λ and $\lambda/2$.

$H^{2+\lambda,1+\lambda/2}(\overline{Q})$ is the space of fuctions $u(x,t)$ continuous for $(x,t) \in \overline{Q}$ together with their derivatives u_{xx}, u_t which satisfy the Hölder conditions in x and t with the corresponding exponents λ and $\lambda/2$.

2. Mathematical statements of one-phase models with unknown convection coefficients

Consider a one-phase quasilinear Stefan problem in the direct statement: to find a function $u(x,t)$ in the domain $\overline{Q} = \{0 \le x \le \xi(t), 0 \le t \le T\}$ and a phase boundary $\xi(t)$ for $0 \le t \le T$ satisfying the equation

$$c(x,t,u)u_t - Lu = f(x,t), \quad (x,t) \in Q, \tag{1}$$

with the boundary condition for $x = 0$, $x = \xi(t)$

$$u|_{x=0} = v(t), \quad 0 < t \le T, \tag{2}$$

$$u|_{x=\xi(t)} = u^*(t), \quad 0 < t \le T, \tag{3}$$

the initial condition

$$u|_{t=0} = \varphi(x), \quad 0 \le x \le l_0, \tag{4}$$

and the conditions on the phase boundary

$$a(x,t,u)u_x + \chi(x,t,u)|_{x=\xi(t)} = -\gamma(x,t,u)|_{x=\xi(t)}\xi_t(t), \quad 0 < t \le T, \tag{5}$$

$$\xi|_{t=0} = l_0, \quad l_0 > 0, \tag{6}$$

where Lu is a uniformly elliptic operator of the form

$$Lu \equiv (a(x,t,u)u_x)_x - b(x,t,u)u_x - d(x,t,u), \tag{7}$$

$a \ge a_{\min} > 0$, b, $c \ge c_{\min} > 0$, d, f, v, u^*, $\gamma \ge \gamma_{\min} > 0$, χ, and φ are known functions, $a_{\min}$, $c_{\min}$, $\gamma_{\min}$, and $l_0 = \text{const} > 0$.

If the function $b(x,t,u)$ in (7) is unknown but the additional information of the solution of the direct Stefan problem (1)–(6) is given at $t = T$

$$u|_{t=T} = g(x), \quad 0 \le x \le l, \quad \xi|_{t=T} = l, \quad l > 0, \tag{8}$$

then the following statement of the inverse restoration problem with final overdetermination arises: to find a function $u(x,t)$ in the domain $\overline{Q}$, a phase boundary $\xi(t)$ for $0 \le t \le T$, and a coefficient $b(x,t,u)$ for $(x,t) \in \overline{Q}$ and $u \in [-M_0, M_0]$ (where $M_0 > 0$ is the constant from the maximum principle for the boundary value problem (1)–(4)) that satisfy conditions (1)–(7) and the additional condition (8).

In what follows, we assume that $b(x,t,u)$ has one of the structures

$$\begin{gathered} b(x,t,u) = p(u)b_0(x,t), \\ b(x,t,u) = p(x,u)b_0(x,t) \end{gathered} \tag{9}$$

where $b_0(x,t)$ is a given function and p is an unknown coefficient.

According to [14], the following theorem formulates requirements on the input data, which imply the assumptions for the corresponding inverse restoration problem.

Theorem 1. *Let the following conditions hold.*

(i) For $(x,t) \in \overline{Q}$, $|u| < \infty$, the functions a, a_x, a_u, b_0, c, d, and f are uniformly bounded, $a \geq a_{\min} > 0$, $c \geq c_{\min} > 0$.

(ii) For $(x,t,u) \in \overline{D} = \overline{Q} \times [-M_0, M_0]$ the function a, its derivatives a_x and a_u, the functions c, d, γ, and χ have continuous x- and u-derivatives and, moreover, are Hölder continuous in t with the exponent $\lambda/2$; the functions b_0 and f have continuous x-derivatives and are Hölder continuous in t with the exponent $\lambda/2$; $\gamma \geq \gamma_{\min} > 0$.

(iii) The functions $v(t)$, $u^*(t)$, and $\varphi(x)$ belong to $H^{1+\lambda/2}[0,T]$ and $H^{2+\lambda}[0,l_0]$, respectively, and satisfy the matching conditions

$$\begin{aligned} c(x,0,\varphi)v_t - L\varphi|_{x=0,t=0} &= f(x,0)|_{x=0}, \\ c(x,0,\varphi)u^*_t - L\varphi|_{x=l_0,t=0} &= f(x,0)|_{x=l_0}. \end{aligned} \tag{10}$$

(iv) The input data provide the nondegeneracy of the domain $\overline{Q}$, i.e., the phase boundary does not intersect the external boundary $x = 0$: $\beta_0 < \xi(t)$ for $0 \leq t \leq T$, where $\beta_0 = const > 0$.

(v) The final function $g(x)$ belongs to $H^{2+\lambda}[0,l]$ and satisfies the matching conditions $g|_{x=0} = v|_{t=T}$, $g|_{x=l} = u^*|_{t=T}$.

Then for any coefficient p from the structure (9) that belongs to the corresponding class

$$p(u) \in C^1[-M_0, M_0], \quad p(x,u) \in C^{1,1}(\overline{\Omega}),$$

$$\overline{\Omega} = [0, \beta_1] \times [-M_0, M_0], \quad \beta_1 = \max_{0 \leq t \leq T} \xi(t),$$

and satisfies the matching conditions (10), the quasilinear Stefan problem in the direct statement (1)–(6) has a unique solution in the Hölder spaces $u(x,t) \in H^{2+\lambda,1+\lambda/2}(\overline{Q})$, $\xi(t) \in H^{1+\lambda/2}[0,T]$ for which the uniform estimates are valid

$$|u|^{2+\lambda,1+\lambda/2}_{\overline{Q}} \leq M, \quad |\xi|^{1+\lambda/2}_{[0,T]} \leq \mathcal{M}, \quad M, \mathcal{M} = const > 0. \tag{11}$$

Theorem 1 allows one to define a solution of the corresponding inverse restoration problem as a collection of functions $\{u(x,t), \xi(t), p(u)\}$ or $\{u(x,t), \xi(t), p(x,u)\}$ that belong to the above-mentioned classes and satisfy relations (1)–(8) in the usual sense. For this ill-posed problem we eximane the conditions under which its solution is uniquely determined.

3. Unique restoration of the convection coefficient $p(u)$

3.1. The proposed approach to the proof. We use a contradiction argument. Assume that $\{u_1(x,t), \xi_1(t), p_1(u)\}$ and $\{u_2(x,t), \xi_2(t), p_2(u)\}$ are two solutions of the inverse problem in the classes $H^{2+\lambda,1+\lambda/2}(\overline{Q}) \times H^{1+\lambda/2}[0,T] \times C^1[-M_0, M_0]$. The functions $\{u_1(x,t), \xi_1(t)\}$ and $\{u_2(x,t), \xi_2(t)\}$ can be treated as the solutions of the direct

Stefan problem (1)–(6) that correspond to the coefficients $p_1(u)$ and $p_2(u)$ in the operator Lu (see (7) and (9)). Therefore, they satisfy estimates (11) in the Hölder classes $H^{2+\lambda,1+\lambda/2}(\overline{Q}) \times H^{1+\lambda/2}[0,T]$.

Before proving that $u_1(x,t) \equiv u_2(x,t)$ in $\overline{Q}$, $\xi_1(t) \equiv \xi_2(t)$ for $0 \le t \le T$, and $p_1(u) \equiv p_2(u)$ for $u \in [-M_0, M_0]$, we make "straightening phase boundary" substitution $y = x\xi^{-1}(t)$. This substitution transforms the phase domain $\overline{Q}$ into a rectangular domain of fixed width $\overline{\Pi} = \{0 \le y \le 1, 0 \le t \le T\}$.

In variables (y,t) the inverse Stefan problem (1)–(8) becomes

$$cu_t - \xi^{-2}(t)(au_y)_y + \xi^{-1}(t)\{pb_0 + cy\xi_t(t)\}u_y + d = f, \quad (y,t) \in \Pi, \tag{12}$$

$$u|_{y=0} = v(t), \quad u|_{y=1} = u^*(t), \quad 0 < t \le T, \tag{13}$$

$$u|_{t=0} = \varphi(yl_0), \quad \xi|_{t=0} = l_0, \quad 0 \le y \le 1, \tag{14}$$

$$\xi^{-1}(t)au_y + \chi|_{y=1} = -\gamma|_{y=1}\xi_t(t), \quad 0 < t \le T, \tag{15}$$

$$u|_{t=T} = g(yl), \quad \xi|_{t=T} = l, \quad 0 \le y \le 1. \tag{16}$$

The coefficients in the equation (12) and in the Stefan condition (15) are the values of the corresponding functions at the point $(y\xi(t), t, u)$. In view of (12)–(16) the differences $\Delta u = u_2 - u_1$, $\Delta\xi = \xi_2 - \xi_1$, and $\Delta p = p_2 - p_1$ satisfy relations that can be represented in the form

$$c\Delta u_t - \xi_2^{-2}(t)(a\Delta u_y)_y + \mathcal{A}\Delta u_y + \mathcal{B}\Delta u$$
$$= \mathcal{C}\Delta\xi(t) + \mathcal{D}\Delta\xi_t(t) - \xi_2^{-1}(t)b_0u_{2y}\Delta p(u_2), \ (y,t) \in \Pi, \tag{17}$$

$$\Delta u|_{y=0} = 0, \quad \Delta u|_{y=1} = 0, \quad 0 < t \le T, \tag{18}$$

$$\Delta u|_{t=0} = 0, \quad 0 \le y \le 1, \tag{19}$$

$$\xi_2^{-1}(t)a\Delta u_y|_{y=1} = -\gamma|_{y=1}\Delta\xi_t(t) + \mathcal{F}|_{y=1}\Delta\xi(t), \quad 0 < t \le T, \quad \Delta\xi|_{t=0} = 0, \tag{20}$$

with additional conditions at $t = T$

$$\Delta u|_{t=T} = 0, \quad 0 \le y \le 1, \quad \Delta\xi|_{t=T} = 0. \tag{21}$$

Here a, b_0, c, γ, etc., are the values of these functions at the point $(y\xi_2(t), t, u_2)$. The coefficients $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{F}$ depend appropriately on u_2, its derivatives u_{2y}, u_{2yy}, and u_{2t}. Moreover, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{F}$ depend appropriately on the y- and u-derivatives of the coefficients in the equation (12) and the Stefan condition (15) at the intermediate point $(y\xi(t), t, u)$ with $\xi(t) = \sigma\xi_1(t) + (1-\sigma)\xi_2(t)$ and $u = \theta u_1 + (1-\theta)u_2$ for $0 < \sigma < 1$ and $0 < \theta < 1$. All these coefficients regarded as functions of (y,t) are in $H^{\lambda,\lambda/2}$ in the domain $\overline{\Pi} = \{0 \le y \le 1, 0 \le t \le T\}$ in view of smoothness conditions (i)–(iii) of Theorem 1 and estimates (11) in the Hölder classes. In particular, the coefficient $\mathcal{A}(y,t)$ has the form $\mathcal{A}(y,t) = \xi^{-1}(t)\{p_2b_0 + cy\xi_t(t) - a_uu_y\}$ and is in $H^{\lambda,\lambda/2}(\overline{\Pi})$ in view of condition (ii) on b_0, c, and a_u, estimates (11) for u_1, u_2, and since $p_2 \in C^1[-M_0, M_0]$. Moreover, y-derivative of $\mathcal{A}(y,t)$ is continuous in $\overline{\Pi}$.

In order to prove that $\Delta u \equiv 0$ in $\overline{\Pi}$, $\Delta\xi \equiv 0$ for $0 \le t \le T$, and $\Delta p \equiv 0$ for $u \in [-M_0, M_0]$, we use the duality principle by analogy with [18], where it was applied for the coefficient inverse problem in a domain with fixed boundary.

3.2. The duality principle and properties of adjoint problems. We remark that the relations (17)–(19) are linear with respect to Δu, $\Delta\xi$, and Δp. This allows one to start with the study of the corresponding boundary value problem for the equation

$$c\Delta u_t - \mathcal{L}\Delta u = -\xi_2^{-1}(t) b_0 u_{2y} \Delta p(u_2), \ (y,t) \in \Pi, \tag{22}$$

$$\mathcal{L}\Delta u \equiv \xi_2^{-2}(t)(a\Delta u_y)_y - \mathcal{A}\Delta u_y - \mathcal{B}\Delta u.$$

Consider the boundary value problem adjoint to (22), (18), (19),

$$(c\psi)_t + \mathcal{L}^*\psi = 0, \quad 0 < y < 1, \quad 0 \le t < T, \tag{23}$$

$$\psi|_{y=0} = 0, \quad \psi|_{y=1} = 0, \quad 0 \le t < T, \tag{24}$$

$$\psi|_{t=T} = \eta(y), \quad 0 \le y \le 1, \tag{25}$$

where $\eta(y)$ is an arbitrary function from $\overset{0}{C}[0,1]$ and

$$\mathcal{L}^*\psi \equiv \xi_2^{-2}(t)(a\psi_y)_y + (\mathcal{A}\psi)_y - \mathcal{B}\psi$$

is the operator adjoint to the operator $\mathcal{L}\Delta u$.

The solution of this linear boundary value problem is defined by $\psi(y,t;\eta)$. Next we investigate the properties of $\psi(y,t;\eta)$.

Lemma 1. *Assume that conditions (i)–(v) of Theorem 1 hold and, moreover, the derivative b_{0x} is in $H^{\lambda,\lambda/2}(\overline{Q})$, the derivative c_t is Hölder continuous in x and t with the corresponding exponents λ and $\lambda/2$, its derivative with respect to u is continuous for $(x,t,u) \in \overline{D}$. Then, for any function $\eta(y) \in \overset{0}{C}[0,1]$, the corresponding solution $\psi(y,t;\eta)$ of the adjoint problem (23)–(25) belongs to $C(\overline{\Pi}) \cap C^{2,1}(\Pi)$ and satisfies the relation*

$$\int_0^T\int_0^1 \psi(y,t;\eta) h(y,t)\, dy\, dt = 0 \quad \forall \eta \in \overset{0}{C}[0,1], \qquad h(y,t) = -\xi_2^{-1}(t) b_0 u_{2y} \Delta p(u_2). \tag{26}$$

Proof. Unique solvability of the problem (23)–(25) in $C(\overline{\Pi}) \cap C^{2,1}(\Pi)$ for any $\eta \in \overset{0}{C}[0,1]$ follows from [19] thanks to the corresponding smoothness of the coefficients in the equation (23); in particular, y-derivative of the coefficient $\mathcal{A}(y,t)$ belongs to $H^{\lambda,\lambda/2}(\overline{\Pi})$.

To prove (26) we consider the expression

$$I = \int_0^T\int_0^1 \psi\{c\Delta u_t - \mathcal{L}\Delta u\}\, dy\, dt + \int_0^T\int_0^1 \Delta u\{(c\psi)_t + \mathcal{L}^*\psi\}\, dy\, dt.$$

On the one hand, from (22) and (23) it follows that

$$I = \int_0^T\int_0^1 \psi(y,t;\eta) h(y,t)\, dy\, dt.$$

On the other hand, integrating by parts and taking into account (18), (19) and (24), (25), and the final condition (21) for $\Delta u|_{t=T}$, we obtain

$$I = \int_0^1 \{c\psi \Delta u\}\Big|_{t=0}^{t=T} dy = 0,$$

which yields the relation (26). Lemma 1 is proved.

It should be noted that the condition $\Delta u|_{t=T} = 0$ is just what $\eta(y)$ in (25) can be an arbitrary function from $\overset{0}{C}[0,1]$. As a result, the adjoint problem (23)–(25) have the same properties as a control problem with a control function in the initial condition. The role of this function is played by $\eta(y)$. The change of variable $t' = T - t$ in (23)–(25) gives a usual control problem for a linear parabolic equation.

The following lemmas show that $\psi(y,t;\eta)$ posses density properties (by analogy with a solution of the control problem).

Lemma 2. *Let the conditions of Lemma 1 be satisfied; in addition, let the derivative a_t be continuous in the domain $\overline{D}$. Then, as the function $\eta(y)$ ranges over the space $\overset{0}{C}[0,1]$, the corresponding set of values $\{\psi(y,t;\eta)|_{t=\tau}\}$ is everywhere dense in $L_2[0,1]$ at any time $t=\tau$; i.e., the relation*

$$\int_0^1 \psi(y,t;\eta)|_{t=\tau}\, w(y)\, dy = 0, \quad 0 < \tau \le T,$$

for some function $w(y) \in \overset{0}{C}[0,1]$ implies that $w(y) = 0$ for $0 \le y \le 1$.

Proof. To establish Lemma 2 we again use the duality principle but now for the problem (23)–(25). Namely, we consider the linear boundary value problem adjoint to (23)–(25) in the domain $\overline{\Pi}_\tau = \{0 \le y \le 1, \tau \le t \le T\}$

$$cz_t - \mathcal{L}z = 0, \quad 0 < y < 1, \quad \tau < t \le T, \tag{27}$$

$$z|_{y=0} = 0, \quad z|_{y=1} = 0, \quad \tau < t \le T, \tag{28}$$

$$z|_{t=\tau} = \theta(y;\tau), \quad 0 \le y \le 1, \tag{29}$$

where the operator $\mathcal{L}z$ has the same form as $\mathcal{L}\Delta u$ and

$$\theta(y;\tau) = \{c(y\xi_2(t), t, u_2)|_{t=\tau}\}^{-1} w(y).$$

Its solution $z(y,t;\tau)$ belongs to $\overset{0}{C}(\overline{\Pi}_\tau) \cap C^{2,1}(\Pi_\tau)$ and is a continuous function of the parameter τ in view of its stability with respect to the input data [19]. For it we obtain the additional final condition $z(y,t;\tau)|_{t=T} = 0$ with the use of the continuous function

$$F(\tau) = \int_\tau^T \int_0^1 z\{(c\psi)_t + \mathcal{L}^*\psi\}\, dy\, dt + \int_\tau^T \int_0^1 \psi\{cz_t - \mathcal{L}z\}\, dy\, dt.$$

In fact, by virtue of (23)–(25) and (27)–(29), $F(\tau)$ can be reduced to the form

$$F(\tau) = \int_0^1 c|_{t=T} z(y, T; \tau)\eta(y)\, dy - \int_0^1 c|_{t=\tau}\theta(y; \tau)\psi(y, \tau; \eta)\, dy = 0 \qquad (30)$$

for any $\eta \in \overset{0}{C}[0, 1]$. From here, taking into account the form of $\theta(y; \tau)$ and the assertion about $w(y)$, we conclude that $z(y, t; \tau)|_{t=T} = 0$ (thanks to the assumption $c \geq c_{\min} > 0$ and density of the space $\overset{0}{C}[0, 1]$ in $L_2[0, 1]$).

This final condition permits one to treat the equation (27) with the conditions (28) as a homogeneous boundary value problem for a linear parabolic equation in inverse time. By smoothness and uniform boundedness in $\overline{\Pi}_\tau$, the coefficients of the equation (27) considered as functions of (y, t) satisfy the requirements [20, 21] that provide the so-called inverse uniqueness property for such a problem. Hence $z(y, t; \tau) \equiv 0$ in $\overline{\Pi}_\tau$ including $t = \tau$; i.e., $\theta(y; \tau) = 0$ and $w(y) = 0$ for $0 \leq y \leq 1$. Thus, the fact that the set $\{\psi(y, t; \eta)|_{t=\tau}\}$ is dense follows from the inverse uniqueness property. The proof of Lemma 2 is completed.

The following result is a generalization of Lemma 2 for an arbitrary time interval $[0, T_0]$, $0 < T_0 \leq T$.

Lemma 3. *Let the conditions of Lemma 2 for the input data hold. Assume that for any function* $\eta \in \overset{0}{C}[0, 1]$, *the corresponding solution* $\psi(y, t; \eta)$ *of the adjoint problem satisfies the relation on some interval* $[0, T_0]$, $0 < T_0 \leq T$,

$$\int_0^{T_0}\int_0^1 \psi(y, t; \eta)\, \alpha(y, t)\, dy\, dt = 0 \quad \forall \eta \in \overset{0}{C}[0, 1], \qquad (31)$$

where $\alpha(y, t)$ *is a function of constant signs with respect to* $t \in [0, T]$ *and, moreover,* $\alpha(y, t)$ *is in* $H^{\lambda, \lambda/2}(\overline{\Pi})$. *Then* $\alpha(y, T_0) = 0$ *for* $0 \leq y \leq 1$.

Proof. Just as in the proof of Lemma 2, consider the problem (27)–(29) in the domain $\overline{\Pi}_\tau$ but for $\theta(y; \tau)$ of the form

$$\theta(y; \tau) = \{c(y\xi_2(t), t, u_2)|_{t=\tau}\}^{-1}\alpha(y, \tau)$$

and for all τ such that $0 \leq \tau \leq T_0$.

The function $F(\tau)$ (see (30)) satisfies the relation

$$\int_0^{T_0} F(\tau)\, d\tau = \int_0^1\int_0^{T_0} z(y, T; \tau)\, d\tau\, c|_{t=T}\eta(y)\, dy - \int_0^{T_0}\int_0^1 \psi(y, \tau; \eta)c|_{t=\tau}\theta(y; \tau)\, dy\, d\tau = 0.$$

In view of the form of $\theta(y; \tau)$ this means (together with (31), the arbitrary choice of the function $\eta(y)$, and positiveness of the coefficient c) that

$$\int_0^{T_0} z(y, T; \tau)\, d\tau = 0, \quad 0 \leq y \leq 1,$$

where the integrand $z(y, T; \tau)$ is the solution of the problem (27)–(29) at the final time $t = T$. By using Green's function $G(y, x, t, \tau)$ [19] for representation of the solution $z(y, t; \tau)$ of this problem, we obtain

$$\int_0^{T_0} z(y, T; \tau)\, d\tau = \int_0^{T_0}\int_0^1 G(y, x, T, \tau)\theta(x; \tau)\, dx\, d\tau = 0, \quad 0 \le y \le 1.$$

We can write this equality in the form

$$\int_0^{T}\int_0^1 G(y, x, T, \tau)\Theta(x; \tau)\, dx\, d\tau = 0, \quad 0 \le y \le 1, \tag{32}$$

where $\Theta(x; \tau) = \begin{cases} \theta(x; \tau) & \text{for } 0 < \tau \le T_0, \\ 0 & \text{for } T_0 < \tau \le T. \end{cases}$

Now we consider the boundary value problem in the domain $\overline{\Pi} = \{0 \le y \le 1, 0 \le t \le T\}$ for the nonhomogeneous equation

$$cZ_t - \mathcal{L}Z = \Theta(y, \tau), \quad 0 < y < 1, \quad 0 < t \le T, \tag{33}$$

$$Z|_{y=0} = 0, \quad Z|_{y=1} = 0, \quad 0 < t \le T, \tag{34}$$

$$Z|_{t=0} = 0, \quad 0 \le y \le 1, \tag{35}$$

and show that its solution $Z(y, t)$ is a smooth function in $\overline{\Pi}$.

In fact, for $0 < y < 1$, $0 < t \le T_0$ we have $\Theta(y, t) = \theta(y, t)$ and $\theta(y, t) \in H^{\lambda, \lambda/2}$, hence $Z(y, t)$ belongs to $C^{2,1}$ for such values of y and t [19]. On the other hand, for $T_0 < t \le T$ the function $\Theta(y, t) = 0$. This means that for $T_0 < t \le T$ $Z(y, t)$ can be represented as a solution $z(y, t; T_0)$ of the boundary value problem in the domain $\overline{\Pi}_{T_0} = \{0 \le y \le 1, T_0 \le t \le T\}$ for the homogeneous equation

$$cz_t - \mathcal{L}z = 0, \quad 0 < y < 1, \quad T_0 < t \le T,$$

with the homogeneous boundary conditions at $y = 0$, $y = 1$, and with the initial condition

$$z|_{t=T_0} = Z(y, T_0), \quad 0 \le y \le 1,$$

where $Z(y, T_0)$ is a solution of the problem (33)–(35) obtained at $t = T_0$. Since $Z(y, T_0) \in \overset{0}{C}[0, 1] \cap C^2(0, 1)$ then $z(y, t; T_0)$ belongs to $\overset{0}{C}(\overline{\Pi}_{T_0}) \cap C^{2,1}(\Pi_{T_0})$ [19]. This allows one to conclude that $Z(y, t)$ also belongs to $\overset{0}{C}(\overline{\Pi}_{T_0}) \cap C^{2,1}(\Pi_{T_0})$ as $Z(y, t)$ coincides with $z(y, t; T_0)$ in this domain. Thus, the solution $Z(y, t)$ of the problem (33)–(35) is continuous everywhere in the domain $\overline{\Pi} = \{0 \le y \le 1, 0 \le t \le T\}$, and $Z(y, t)$ belongs to $C^{2,1}$ in the above-mentioned subdomains of this domain.

Since the equality (32) is a representation of this solution at the final time $t = T$ [19], then from (32) it follows that $Z(y, T) = 0$ for $0 \le y \le 1$. But $Z(y, T) = z(y, T; T_0)$, hence $z(y, T; T_0)$ is also equal to 0 for $0 \le y \le 1$. Thus, in the domain $\overline{\Pi}_{T_0}$ the solution of the homogeneous equation with the homogeneous boundary conditions satisfies the final condition $z(y,t;T_0)|_{t=T} = 0$ for $0 \le y \le 1$. Just as in the proof of Lemma 2 we can use results of [20, 21] on the inverse uniqueness property; i.e., $z(y,t;T_0) \equiv 0$ in $\overline{\Pi}_{T_0}$. Then it follows from the initial condition $z|_{t=T_0} = Z(y,T_0)$ that $Z(y,T_0) = 0$ for $0 \le y \le 1$. But $Z(y,T_0)$ satisfies the nonhomogeneous equation (33) with the right hand side $\Theta(y,t) = \theta(y,t)$ for $t = T_0$. Hence, $\theta(y,T_0) = 0$ for $0 \le y \le 1$. This means (see the form of the function $\theta(y,t)$) that $\alpha(y,T_0) = 0$ for $0 \le y \le 1$. Lemma 3 is proved.

3.3. Conditions of unique restoration of $p(u)$. The density properties for the adjoint problem (23)–(25) established with the help of the duality principle permit one to investigate the uniqueness of a solution of the inverse restoration problem (1)–(8) with an unknown coefficient $p(u)$.

Theorem 2. *Let the following conditions be satisfied.*

1. *There hold assumptions of Theorem 1 for the input data; in addition, the coefficient b_0 is positive for $(x,t) \in \overline{Q}$, its derivative b_{0x} is in $H^{\lambda,\lambda/2}(\overline{Q})$, the derivative a_t is in $C(\overline{D})$, the derivative c_t is Hölder continuous in x and t with the corresponding exponents λ and $\lambda/2$, its derivative with respect to u is continuous for $(x,t,u) \in \overline{D}$; the derivative of the final function $g(x)$ is a sign-definite function: $|g_x(x)| > 0$ for $0 \le x \le l$.*

2. *There exists a solution $\{u(x,t), \xi(t), p(u)\}$ of the considered inverse restoration problem possessing the properties*

$$u(x,t) \in H^{2+\lambda,1+\lambda/2}(\overline{Q}), \quad p(u) \in C^1[-M_0,M_0], \quad 0 < \lambda < 1,$$

$$u(x,t)_x \text{ is a function of constant signs with respect to } t \in [0,T],$$

$$\xi(t) \in H^{1+\lambda/2}[0,T], \quad 0 < \beta_0 < \xi(t) \le \beta_1 \text{ for } 0 \le t \le T,$$

and satisfying the relations (1)–(7), the final observation (8), and the matching conditions (10).

Then this solution is unique in the mentioned classes of smooth functions under one of the following conditions

(j) $p(u)$ is defined for $u \in [-M_0, g_{\min})$ and $u \in (g_{\max}, M_0]$, where $g_{\min} = \min_{0\le x\le l} g(x)$ and $g_{\max} = \max_{0\le x\le l} g(x)$,

(jj) $p(u)$ is an analytic function for $u \in (-M_0, M_0)$.

Proof. To prove this theorem, first we consider the equation (22) with conditions (18), (19) and the corresponding adjoint boundary value problem (23)–(25). The assumptions on the input data allow one to apply Lemma 3 to the integral relation (26) of Lemma 1 with $\alpha(y,t) = h(y,t)$, where we suppose that Δp is a function of constant signs with respect to $u \in [M_0, M_0]$ (see the form of the function $h(y,t)$). Hence, we conclude that

$$\{\xi_2^{-1}(t) b_0 u_{2y} \Delta p(u_2)\}\Big|_{t=T} = 0, \quad 0 \le y \le 1.$$

Since $\xi(t)|_{t=T} = l$ then taking into account this fact and the inequalities $b_0(yl, T) > 0$ and $|g_y(yl)| > 0$ for $0 \le y \le 1$, we obtain $\Delta p(g(yl)) = 0$ for $0 \le y \le 1$. Since the function $g(x)$ is continuous for $0 \le x \le l$, we have $\Delta p(g) = 0$ for $g \in [g_{\min}, g_{\max}]$. Under either of assumptions (j) and (jj), this means that $\Delta p(u) = 0$ for $u \in [-M_0, M_0]$. Then the equation (22) together with the conditions (18), (19) implies $\Delta u(x,t) \equiv 0$ in $\overline{Q}$ (in variables (x,t)) [19].

Now we return to the equation (17) and consider its other linear part, namely

$$c\Delta u_t - \xi_2^{-2}(t)(a\Delta u_y)_y + \mathcal{A}\Delta u_y + \mathcal{B}\Delta u = \mathcal{C}\Delta\xi(t) + \mathcal{D}\Delta\xi_t(t), \quad (y,t) \in \Pi. \tag{36}$$

But from the equation (36) and the relations (18)–(20) it follows that $\Delta u(x,t) \equiv 0$ in $\overline{Q}$ (in variables (x,t)), $\Delta\xi(t) \equiv 0$ for $0 \le t \le T$ since the direct quasilinear Stefan problem (1)–(7) with the coefficient $b = p(u)b_0(x,t)$ has a unique solution (see [14]).

Thus, results obtained for the equations (22) and (36) with the corresponding boundary and initial conditions allow one to complete the proof of Theorem 2.

4. Unique restoration of the convection coefficient $p(x,u)$

Conditions for the uniqueness of the solution $\{u(x,t), \xi(t), p(x,u)\}$ of inverse restoration problem (1)–(8) with the unknown convection coefficient dependent on the temperature u and the spatial variable x are established by the following theorem.

Theorem 3. *Let assumption 1 of Theorem 2 be satisfied. In addition, suppose that there exists a solution $\{u(x,t), \xi(t), p(x,u)\}$ satisfying the relations (1)–(7), the final observation (8), and the matching conditions (10) and having the properties*

$$u(x,t) \in H^{2+\lambda,1+\lambda/2}(\overline{Q}), \quad \xi(t) \in H^{1+\lambda/2}[0,T], \quad p(x,u) \in C^{1,1}(\overline{\Omega}), \quad 0 < \lambda < 1,$$

$$u(x,t)_x \textit{ is a function of constant signs with respect to } t \in [0,T],$$

$$0 < \beta_0 < \xi(t) \le l_0 = \beta_1 \textit{ for } 0 \le t \le T, \quad \overline{\Omega} = [0,\beta_1] \times [-M_0, M_0].$$

Then this solution is unique in the mentioned classes of smooth functions under one of the following conditions

(jjj) $p(x,u)$ is defined in $\overline{\Omega}$ outside the domain $\{(x,u) : 0 \le x \le l, g_{\min} \le u \le g_{\max}\}$, where $g_{\min} = \min_{0 \le x \le l} g(x)$ and $g_{\max} = \max_{0 \le x \le l} g(x)$,

(jv) $p(x,u)$ is an analytic function in the domain $\overline{\Omega}$.

The proof of these claims is similar to that of Theorem 2. In particular, an analog of the equation (17) is given by the equation

$$c\Delta u_t - \xi_2^{-2}(t)(a\Delta u_y)_y + \mathcal{A}\Delta u_y + \mathcal{B}\Delta u$$

$$= \mathcal{C}\Delta\xi(t) + \mathcal{D}\Delta\xi_t(t) - \xi_2^{-1}(t)b_0 u_{2y}\Delta p(y\xi_2(t), u_2), \ (y,t) \in \Pi. \tag{37}$$

Hence, the corresponding form of the equation (22) becomes

$$c\Delta u_t - \mathcal{L}\Delta u = -\xi_2^{-1}(t)b_0 u_{2y}\Delta p(y\xi_2(t), u_2), \ (y,t) \in \Pi. \tag{38}$$

Next, taking into account the assumptions of Theorem 3 on the input data and the solution of this inverse problem, we can apply Lemma 3 with $\alpha(y,t) = h(y,t)$ to the integral relation (26) of Lemma 1. Now the function $h(y,t)$ has the form

$$h(y,t) = -\xi_2^{-1}(t) b_0 u_{2y} \Delta p(y\xi_2(t), u_2),$$

and is a function of constant signs with respect to $t \in [0,T]$. This claim is valid if we take into account, in particular, that $y\xi_2(t) \in [0, l_0]$ for any $t \in [0,T]$ and Δp is a function of constant signs with respect to $u \in [M_0, M_0]$ (according to our supposition). This leads to

$$\{\xi_2^{-1}(t) b_0 u_{2y} \Delta p(y\xi_2(t), u_2)\}|_{t=T} = 0, \quad 0 \le y \le 1.$$

From here it follows that $\Delta p(yl, g(yl)) = 0$ for $0 \le y \le 1$ since $\xi(T) = l$, $b_0(yl,T) > 0$, and $|g_y(yl)| > 0$ for $0 \le y \le 1$. This, together with the continuity of the final function $g(x)$, implies that $\Delta p(x,g) \equiv 0$ for $0 \le x \le l$, $g \in [g_{\min}, g_{\max}]$. Hence, any of the assumptions (jjj) and (jv) allows one to conclude that $\Delta p(x,u) \equiv 0$ in the entire domain $\overline{\Omega}$. But this means that the equation (38) with the conditions (18), (19) have a unique solution $\Delta u(x,t) \equiv 0$ in $\overline{Q}$ (in variables (x,t)) [19].

Investigation of the other linear part of the equation (37) completely repeats the corresponding claims for the equation (36) and implies identities $\Delta u(x,t) \equiv 0$ in $\overline{Q}$ (in variables (x,t)), $\Delta\xi(t) \equiv 0$ for $0 \le t \le T$ since the direct quasilinear Stefan problem (1)–(7) with the coefficient $b = p(x,u) b_0(x,t)$ has a unique solution (see [14]).

This completes the proof of Theorem 3 on uniqueness of the solution $\{u(x,t), \xi(t), p(x,u)\}$.

5. Admissible solutions of the inverse restoration problem

The function spaces chosen for the input data and the solution $\{u, \xi, p\}$ of the considered inverse problems are natural in the sense that they are associated with the exact differential dependences in Hölder classes for the corresponding direct statement of the one-phase quasilinear Stefan problem (1)–(7) [14]. However, if the set of admissible solutions is expanded by assuming that the desired coefficient p in (9) also depends on the variable t, the uniqueness property may be lost. This is illustrated by the following examples.

Example 1. Two function sets

$$\begin{cases} u_1(x,t) &= x(2-t), \\ \xi_1(t) &= 2 - t^2, \\ p_1(t,u) &= u + t^2 + 0.5, \end{cases}$$

$$\begin{cases} u_2(x,t) &= x(2-t^2), \\ \xi_2(t) &= 2 - t, \\ p_2(t,u) &= u + t^2 + t, \end{cases}$$

are solutions of the following inverse restoration problem in the one-phase domain $\overline{Q} = \{0 \le x \le \xi(t), 0 \le t \le 1\}$:

$$u_t - u_{xx} + xp(t,u)u_x - 2xu = 2xt^2, \quad (x,t) \in Q,$$

$$u|_{x=0} = 0, \quad u|_{x=\xi(t)} = (2-t^2)(2-t), \quad 0 < t \le 1,$$
$$u|_{t=0} = 2x, \quad 0 \le x \le 2, \quad \xi|_{t=0} = 2,$$
$$u|_{t=1} = x, \quad 0 \le x \le 1, \quad \xi|_{t=1} = 1,$$
$$u_x + \chi(x,t)|_{x=\xi(t)} = \xi_t(t), \quad 0 < t \le 1,$$

where the function $\chi(x,t)|_{x=\xi(t)}$ has the form

$$\chi(x,t)|_{x=\xi(t)} = \xi(t)\frac{1-2t}{t(1-t)} + \frac{2t(2-t)-(2-t^2)}{t(1-t)} - 2.$$

Example 2. Two function sets

$$\begin{cases} u_1(x,t) &= x(2-t)(x+t^2), \\ \xi_1(t) &= 2-t^2, \\ p_1(x,t,u) &= \frac{u+2(t-2)+x(4t-3t^2-x)}{(t-2)(2x+t^2)}, \end{cases}$$

$$\begin{cases} u_2(x,t) &= x(2-t^2)(x+t), \\ \xi_2(t) &= 2-t, \\ p_2(x,t,u) &= \frac{u+2(t^2-2)+x(2-3t^2-2xt)}{(t^2-2)(2x+t)}, \end{cases}$$

are solutions of the following inverse restoration problem in the one-phase domain $\overline{Q} = \{0 \le x \le \xi(t), 0 \le t \le 1\}$:

$$u_t - u_{xx} + p(x,t,u)u_x + u = 0, \quad (x,t) \in Q,$$
$$u|_{x=0} = 0, \quad u|_{x=\xi(t)} = 2(2-t^2)(2-t), \quad 0 < t \le 1,$$
$$u|_{t=0} = 2x^2, \quad 0 \le x \le 2, \quad \xi|_{t=0} = 2,$$
$$u|_{t=1} = x(x+1), \quad 0 \le x \le 1, \quad \xi|_{t=1} = 1,$$
$$u_x + \chi(x,t)|_{x=\xi(t)} = \xi_t(t), \quad 0 < t \le 1,$$

where the function $\chi(x,t)|_{x=\xi(t)}$ has the form

$$\chi(x,t)|_{x=\xi(t)} = (2t-1)\frac{\xi(t)-(2-t)}{t(t-1)} - 1 - (\xi(t)+2)(4-t^2-t-\xi(t)).$$

Therefore, the function sets in the corresponding statements of the inverse restoration problems in the one-phase domain

$$\{u(x,t), \xi(t), p(u)\} \in H^{2+\lambda,1+\lambda/2}(\overline{Q}) \times H^{1+\lambda/2}[0,T] \times C^1[-M_0, M_0],$$
$$\{u(x,t), \xi(t), p(x,u)\} \in H^{2+\lambda,1+\lambda/2}(\overline{Q}) \times H^{1+\lambda/2}[0,T] \times C^{1,1}(\overline{\Omega})$$

form natural sets of admissible solutions preserving the uniqueness property. The extension of these sets by the inclusion of nonlinear convection coefficients also depending on time t leads to the possible failure of this property.

6. Conclusion

The mathematical models of one-phase heat transform processes with unknown temperature-dependent convection coefficients are investigated. The following results of this analysis can be formulated.

1. The statements of the corresponding inverse problems on the identification of nonlinear convection coefficients are justified under the assumption that additional information is given in the form of final observation of the temperature distribution and the phase boundary position. The choice of function spaces for the input data and the solution of such inverse problems relies on unique solvability of the corresponding direct Stefan problems in Hölder classes.

2. For these statements the conditions ensuring the uniqueness of the smooth solution are obtained. The corresponding proof relies on the "straightening phase boundaries" substitution, the next application of the duality principle, and the study of the density properties for the corresponding adjoint problems.

3. The sets of admissible solutions of the inverse restoration problems preserving the uniqueness property are indicated. The corresponding examples show that this property may be lost if the desired convection coefficient depends not only on the temperature and the spatial variable but also on the time.

Investigation of uniqueness property for the inverse restoration problems is important both for the mathematical modeling and numerical solving complicated nonstationary processes and for theory of free boundary problems for parabolic equations.

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