DISCONTINUOUS SOLUTIONS FOR THE SHORT-PULSE MASTER MODE-LOCKING EQUATION

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Abstract. The short-pulse master mode-locking equation is a model for ultrafast pulse propagation in a mode-locked laser cavity in the few-femtosecond pulse regime, that is a nonlinear evolution equation. In this paper, we prove the wellposedness of the Cauchy problem associated with this equation within a class of discontinuous solutions.

1. Introduction

In recent years, the ultrafast science has experimentally crossed the threshold from the routine and direct laser production of few-cycle optical pulses in the $5-8$ fs regime [29, 37, 38], to subfemtosecond pulses in the range of hundreds of attoseconds [35, 36], leading to the hitherto unexplored field of attosecond physics [23, 57, 58].

Such short pulse durations and broad spectral content have opened up vast new possibilities for exploring the fundamental nature of atomic and molecular physics at the fastest time scales, including molecular vibrations, chemical reactions, and light-matter interactions. Indeed, even single-electron transition events can now be captured [35] and an absolute measure of time potentially established [24, 25].

Yet the achievement of attosecond pulses in experiments also highlights the difficulties in measurement and interpretation, especially as the optical field information is interwoven with the physics of the interaction. Thus, theoretical and experimental methods are needed to help guide the understanding of attosecond science.

A first method is based on the master mode-locking (MML) equation. It was developed by Haus [33, 34] and has dominated mode-locking theory for many years.

The Haus theory is fundamentally based upon a center-frequency expansion of the electric field and a derivation of an envelope approximation of the nonlinear Schrödinger equation (NLS) type. Inherent in the model is the assumption that the envelope is slow in comparison to the underlying fast carrier.

In the case of the pulses which contain only a few cycles of the carrier, this approximation fails to hold, even if higher-order terms are incorporated into the NLS-based description. Regardless, the NLS-based approach has been shown to work quantitatively beyond its expected breakdown, into the tens of femtoseconds regime, and has been used extensively for modeling supercontinuum generation [28].

However, when pushed to the extreme of a few femtosecond pulses, the NLS description becomes suspect, and a theory not founded upon a center-frequency expansion is required.

As an alternative to the Haus model, Farnum and Kutz [30, 31, 32] develop a model to describe the pulses like those described in [29, 35, 36, 37, 38] without relying on center-frequency expansion.

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The Farnum and Kutz model is based on a short-pulse master mode-locking (SPMML) theory, which derives from the Maxwell equations [1, 2, 4, 5, 39, 50, 51, 54, 59]. In such theories, it is assumed that the propagation occurs for a broadband pulse so that the center-frequency expansion is circumvented.

From a mathematical point of view, the Farnum and Kutz model is formulated in terms of the following equation:

$$\partial_x \left(\partial_t u + a\partial_x u^3 + bu + cu^3\right) = \gamma u, \quad a,\, b,\, c,\, \gamma \in \mathbb{R}, \tag{1.1}$$

also known as SPMML equation, where $u(t, x)$ is the electric field amplitude, t is the time, x is distance propagated in the laser cavity [54], b is the linear attenuation, c is a cubic gain term giving rise to intensity discrimination (saturable absorption) [33, 34] and γ is a real parameter [39, 54].

If $b = c = 0$, (1.1) reads

$$\partial_x \left(\partial_t u + a\partial_x u^3\right) = \gamma u. \tag{1.2}$$

It was introduced by Kozlov and Sazonov [39] as a model equation describing the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media, and by Schäfer and and Wayne [54] as a model equation describing the propagation of ultra-short light pulses in silica optical fibers. Hence, with respect to Kozlov, Sazonov, Schäfer and and Wayne, Farnum and Kutz take into consideration the linear attenuation and the saturable absorption. Moreover, [42, 43, 44] show that (1.2) is a non-slowly-varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons.

If we take $c = 0$ in (1.1), we have the short pulse equation with linear attenuation, which is deduced in [63].

It also is interesting to remind that equation (1.2) was proposed earlier in [48] in the context of plasma physic and that the similar equations describe the dynamics of radiating gases [41, 56]. In [62], the authors deduce (1.2) to describe the short pulse propagation in nonlinear metamaterials characterized by a weak Kerr-type nonlinearity in their dielectric response. Moreover, [3, 6, 53, 52] show that (1.2) is a particular Rabelo equation which describes pseudospherical surfaces.

In literature, we have the following generalization of (1.2):

$$\partial_x \left(\partial_t u + \kappa\partial_x u^3 - \beta\partial_{xxx}^3 u\right) = \gamma u. \tag{1.3}$$

It was derived by Costanzino, Manukian and Jones [22] in the context of the nonlinear Maxwell equations with high-frequency dispersion. Kozlov and Sazonov [39] show that (1.3) is an more general equation than (1.2) to describe the nonlinear propagation of optical pulses of a few oscillations duration in dielectric media.

Recently, wellposedness results for the Cauchy problem of (1.2) are proven in the context of energy spaces (see [26, 49, 60]). A similar result is proven in [10, 11] in the context of the entropy solution, while, in [13, 14, 18], the wellposedness of the homogeneous initial boundary value problem is studied. Finally, the convergence of a finite difference scheme is studied in [21].

Moreover, mathematical properties of (1.3) are studied in many different contexts, including the local and global wellposedness in energy spaces [22, 49] and stability of solitary waves [22, 46]. Observe that, letting $\beta \to 0$ in (1.3), we obtain (1.2). Hence, following [8, 45, 55], in [11, 15], the convergence of the solution of (1.3) to the unique entropy solution of (1.2) is proven.

In this papaer, we assume that $c > 0$ and we write $c = \kappa^2$. Therefore, (1.1) reads

$$\partial_x \left(\partial_t u + a\partial_x u^3 + bu + \kappa^2 u^3\right) = \gamma u. \tag{1.4}$$

We are interested in the Cauchy problem for this equation, thus we augment (1.4) with the initial condition

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}, \tag{1.5}$$

on which assume that

$$u_0(x) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}), \quad \int_{\mathbb{R}} u_0(x)dx = 0. \tag{1.6}$$

Following [9, 10, 11], on the function

$$P_0(x) = \int_{-\infty}^{x} u_0(y)dy, \tag{1.7}$$

we assume that

$$\|P_0\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left(\int_{-\infty}^{x} u_0(y)dy\right)^2 dx < \infty. \tag{1.8}$$

Integrating (1.4) in $(0,x)$, we gain the integro-differential formulation of (1.4) and (1.6).

$$\begin{cases} \partial_t u + a\partial_x u^3 + bu + \kappa^2 u^3 = \gamma \displaystyle\int_0^x u dy, & t \geq 0,\, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.9}$$

that is equivalent to

$$\begin{cases} \partial_t u + a\partial_x u^3 + bu + \kappa^2 u^3 = \gamma P, & t \geq 0,\, x \in \mathbb{R}, \\ \partial_x P = u, & t \geq 0,\, x \in \mathbb{R}, \\ P(t,0) = 0, & t \geq 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.10}$$

One of the main issues in the analysis of (1.10) is that the equation is not preserving the L^1 norm, as a consequence the nonlocal source term P and the solution u are a priori only locally bounded. Indeed, from (1.9) and (1.10) is clear that we cannot have any L^∞ bound without an L^1 bound. Since we are interested in the bounded solutions of (1.4), some assumptions on the decay at infinity of the initial condition u_0 are needed (see (1.8)).

The unique useful conserved quantities are

$$t \longmapsto \int u(t,x)dx = 0, \qquad t \longmapsto \int u^2(t,x)dx. \tag{1.11}$$

In the sense that if $u(t,\cdot)$ has zero mean at time $t = 0$, then it will have zero mean at any time $t > 0$. In addition, the L^2 norm of $u(t,\cdot)$ is constant with respect to t. Therefore, we require that initial condition u_0 belongs to $L^2 \cap L^\infty$ and has zero mean.

Due to the regularizing effect of the P equation in (1.10), we have that

$$u \in L^\infty((0,T)\times\mathbb{R}) \Longrightarrow P \in L^\infty((0,T);W^{1,\infty}(\mathbb{R})), \quad T > 0. \tag{1.12}$$

Following [9, 10, 11], we give the following definition of solution.

Definition 1.1. *We say that $u \in L^\infty((0,T)\times\mathbb{R})$ is an entropy solution of the initial value problem* (1.10) *and* (1.5) *if*

- *i) u is a distributional solution of* (1.10)*;*
- *ii) for every convex function $\eta \in C^2(\mathbb{R})$ the entropy inequality*

$$\partial_t \eta(u) + \partial_x q(u) + b\eta'(u)u + \kappa^2\eta'(u)u^3 - \gamma\eta'(u)P \leq 0, \qquad q(u) = 3a\int^u \xi^2\eta'(\xi)\,d\xi, \tag{1.13}$$

holds in the sense of distributions in $(0,\infty)\times\mathbb{R}$.

The main result of this paper is the following theorem.

Theorem 1.1. *Assume* (1.6) *and* (1.8)*. The initial value problem* (1.10) *and* (1.5) *possesses an unique entropy solution u in the sense of Definition1.1. Moreover, if u and v are two entropy solutions* (1.10) *and* (1.5) *in the sense of Definition 1.1, the following inequality holds*

$$\|u(t,\cdot)-v(t,\cdot)\|_{L^1(-R,R)} \le e^{C(T)t}\|u(0,\cdot)-v(0,\cdot)\|_{L^1(-R-C(T)t,R+C(T)t)}, \tag{1.14}$$

for almost every $0<t<T$, $R>0$, and some suitable constant $C(T)>0$.

The paper is organized as follows. In Section 2, we prove several a priori estimates on a vanishing viscosity approximation of (1.10). Those play a key role in the proof of our main result, that is given in Section 3.

2. Vanishing viscosity approximation

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.10).

Fix $\varepsilon>0$ a small number and let $u_\varepsilon=u_\varepsilon(t,x)$ be the unique classical solution of the following mixed problem [16, 19]:

$$\begin{cases} \partial_t u_\varepsilon + a\partial_x u_\varepsilon^3 + bu_\varepsilon + \kappa^2 u_\varepsilon^3 = \gamma P_\varepsilon + \varepsilon\partial_{xx}^2 u_\varepsilon, & t>0,\ x\in\mathbb{R},\\ \partial_x P_\varepsilon = u_\varepsilon, & t>0,\ x\in\mathbb{R},\\ P_\varepsilon(t,0)=0, & t>0,\ x\in\mathbb{R},\\ u_\varepsilon(0,x)=u_{\varepsilon,0}(x), & x\in\mathbb{R}, \end{cases} \tag{2.1}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that

$$\begin{aligned} &\|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})} \le \|u\|_{L^\infty(\mathbb{R})}, \quad \|u_{\varepsilon,0}\|_{L^2(\mathbb{R})} \le \|u_0\|_{L^2(\mathbb{R})},\\ &\|u_{\varepsilon,0}\|_{L^4(\mathbb{R})} \le \|u_0\|_{L^4(\mathbb{R})}, \quad \int_{\mathbb{R}} u_{\varepsilon,0}(x)=0,\\ &\|P_{\varepsilon,0}\|_{L^2(\mathbb{R})} \le \|P_0\|_{L^2(\mathbb{R})}, \quad \int_{\mathbb{R}} P_{\varepsilon,0}(x)u_{\varepsilon,0}^3(x)dx \le C_0, \end{aligned} \tag{2.2}$$

where C_0 is a constant independent on ε.

Let us prove some a priori estimates on u_ε and P_ε, denoting with C_0 the constants which depend only on the initial data, and $C(T)$ the constants which depend also on T. Arguing as in [7, Lemmas 2.1], or [9, Lemmas 2.1], we have the following result.

Lemma 2.1. *Let us suppose that*

$$P_\varepsilon(t,-\infty)=0,\quad t\ge 0,\quad (\textit{or } P_\varepsilon(t,\infty)=0). \tag{2.3}$$

Then, the following statements are equivalent

$$\int_{\mathbb{R}} u_\varepsilon(t,x)dx = 0,\quad t\ge 0, \tag{2.4}$$

$$\begin{aligned} &\frac{d}{dt}\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^2 + \kappa^2\|u_\varepsilon(t,\cdot)\|_{L^4(\mathbb{R})}^4\\ &\qquad + 2\varepsilon\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^2 = 2b\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^2,\quad t>0. \end{aligned} \tag{2.5}$$

Lemma 2.2. *For each $t\in[0,\infty)$,* (2.3) *and* (2.4) *hold. In particular, fixed $T\ge t\ge 0$, we have that*

$$\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}^2 + \kappa^2 e^{|b|t}\int_0^t e^{-|b|s}\|u_\varepsilon(s,\cdot)\|_{L^4(\mathbb{R})}^4 ds$$

$$+ 2\varepsilon e^{|b|t}\int_0^t e^{-|b|s}\left\|\partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T). \tag{2.6}$$

Proof. Arguing as in [7, Lemmas 2.2], we have (2.3), and (2.4). Lemma 2.1 says that (2.5) also hold. Therefore, we get

$$\begin{aligned}\frac{d}{dt}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ \kappa^2\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\\ &+ 2\varepsilon\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le 2|b|\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Fixed $T > 0$, the Gronwall Lemma and (2.2) give

$$\begin{aligned}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ \kappa^2 e^{2|b|t}\int_0^t e^{-2|b|s}\left\|u_\varepsilon(s,\cdot)\right\|^4_{L^4(\mathbb{R})} ds\\ &+ 2\varepsilon e^{2|b|t}\int_0^t e^{-2|b|s}\left\|\partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le \left\|u_0\right\|_{L^2(\mathbb{R})} e^{|b|t} \le C(T),\end{aligned}$$

that is (2.6). □

Lemma 2.3. *Fix $T > 0$. There exists a constant $C(T) > 0$, independent on ε, such that*

$$\begin{aligned}&\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + 4\kappa^2\int_0^t \left\|u_\varepsilon(s,\cdot)\right\|^6_{L^6(\mathbb{R})} ds\\ &\quad + 12\varepsilon\int_0^t \left\|u_\varepsilon(s,\cdot)\partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T)\left(1 + \left\|P_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}\right).\end{aligned} \tag{2.7}$$

Proof. Let $0 \le t \le T$. Multiplying (2.1) by $4u_\varepsilon^3$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} =& 4\int_{\mathbb{R}} u_\varepsilon^3 \partial_t u_\varepsilon dx\\ =& -12a\int_{\mathbb{R}} u_\varepsilon^5\partial_x u_\varepsilon dx - 4b\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} - 4\kappa^2\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\ &+ 4\gamma\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx + 4\varepsilon\int_{\mathbb{R}} u_\varepsilon^3\partial^2_{xx}u_\varepsilon dx\\ =& -4b\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} - 4\kappa^2\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\ &+ 4\gamma\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx - 12\varepsilon\left\|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{d}{dt}\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} &+ 4\kappa^2\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})} + 12\varepsilon\left\|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &= -4b\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + 4\gamma\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx\\ &\le 4|b|\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + 4|\gamma|\int_{\mathbb{R}} |P_\varepsilon||u_\varepsilon|^3 dx.\end{aligned} \tag{2.8}$$

Due to (2.6) and the Young inequality,

$$\begin{aligned}4|\gamma|\int_{\mathbb{R}} |P_\varepsilon||u_\varepsilon|^3 dx &\le 4|\gamma|\left\|P_\varepsilon\right\|_{L^\infty(0,T)\times\mathbb{R})}\int_{\mathbb{R}} |u_\varepsilon|^3 dx\\ &\le 2|\gamma|\left\|P_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2|\gamma|\left\|P_\varepsilon\right\|_{L^\infty(0,T)\times\mathbb{R})}\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\\ &\le C(T)\left\|P_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})} + +2|\gamma|\left\|P_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})}.\end{aligned}$$

Consequently, by (2.8),

$$\begin{aligned}\frac{d}{dt}\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})}+4\kappa^2\|u_\varepsilon(t,\cdot)\|^6_{L^6(\mathbb{R})}+12\varepsilon\|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ \le 4|b|\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})}+C(T)\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\\ +2|\gamma|\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})}.\end{aligned}$$

It follows from (2.2), (2.5) and an integration on $(0,t)$ that

$$\begin{aligned}&\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})}+4\kappa^2\int_0^t\|u_\varepsilon(s,\cdot)\|^6_{L^6(\mathbb{R})}ds+12\varepsilon\int_0^t\|u_\varepsilon(s,\cdot)\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\ &\le C_0 4|b|\int_0^t\|u_\varepsilon(s,\cdot)\|^4_{L^4(\mathbb{R})}ds+C(T)\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}t\\ &\quad+2|\gamma|\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\|u_\varepsilon(s,\cdot)\|^4_{L^4(\mathbb{R})}ds\\ &\le C_0 4|b|e^{2|b|t}\int_0^t e^{-2|b|s}\|u_\varepsilon(s,\cdot)\|^4_{L^4(\mathbb{R})}ds+C(T)\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\\ &\quad+2|\gamma|\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}e^{2|b|t}\int_0^t e^{-2|b|s}\|u_\varepsilon(s,\cdot)\|^4_{L^4(\mathbb{R})}ds\\ &\le C(T)\left(1+\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\right),\end{aligned}$$

which give (2.7). □

Lemma 2.4. *Fix $T>0$. For each $t\in[0,T]$, we have that*

$$\int_{-\infty}^0 P_\varepsilon(t,x)dx\le\frac{a}{\gamma}u_\varepsilon^3(t,0)-\frac{\varepsilon}{\gamma}\partial_x u_\varepsilon(t,0)+C(T)\sqrt{\left(1+\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\right)},\tag{2.9}$$

$$\int_0^{\infty} P_\varepsilon(t,x)dx\le-\frac{a}{\gamma}u_\varepsilon^3(t,0)+\frac{\varepsilon}{\gamma}\partial_x u_\varepsilon(t,0)+C(T)\sqrt{\left(1+\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\right)}.\tag{2.10}$$

Moreover,

$$\int_{\mathbb{R}}P_\varepsilon(t,x)dx=-\frac{\kappa^2}{\gamma}\int_{\mathbb{R}}u_\varepsilon^3(t,x)dx,\qquad t\ge 0.\tag{2.11}$$

Proof. We begin by observing that, integrating the second equation of (2.1) on $(0,x)$, we have

$$P_\varepsilon(t,x)=\int_0^x u_\varepsilon(t,y)dy.\tag{2.12}$$

Consequently, by (2.3),

$$\int_0^{-\infty}u_\varepsilon(t,x)dx=0.\tag{2.13}$$

Differentiating (2.13) with respect to t, we obtain

$$\frac{d}{dt}\int_0^{-\infty}u_\varepsilon(t,x)dx=\int_0^{-\infty}\partial_t u_\varepsilon(t,x)dx=0.\tag{2.14}$$

Integrating the first equation on $(0,x)$, we have that

$$\begin{aligned}\gamma\int_0^x P_\varepsilon(t,y)dy=&\int_0^x \partial_t u_\varepsilon(t,y)dy+au_\varepsilon^3(t,x)-au_\varepsilon(t,0)-\varepsilon\partial_x u_\varepsilon(t,x)\\&+\varepsilon\partial_x u_\varepsilon(t,0)-bP_\varepsilon(t,x)-\kappa^2\int_0^x u_\varepsilon^3(t,x)dy.\end{aligned}\tag{2.15}$$

It follows from (2.3) and the regularity of u_ε that

$$\lim_{x\to-\infty}\left(+au_\varepsilon^3(t,x)-\varepsilon\partial_x u_\varepsilon(t,x)-bP_\varepsilon(t,x)\right)=0.\tag{2.16}$$

Consequently, by (2.3), (2.14), (2.15) and (2.16),

$$\gamma\int_0^{-\infty} P_\varepsilon(t,x)dx=-au_\varepsilon(t,0)+\varepsilon\partial_x u_\varepsilon(t,0)-\kappa^2\int_0^{-\infty}u_\varepsilon^3(t,x)dx,$$

that is

$$\int_{-\infty}^0 P_\varepsilon(t,x)dx=\frac{a}{\gamma}u_\varepsilon(t,0)-\frac{\varepsilon}{\gamma}\partial_x u_\varepsilon(t,0)-\frac{\kappa^2}{\gamma}\int_{-\infty}^0 u_\varepsilon^3(t,x)dx\tag{2.17}$$

Due to (2.6), (2.7) and the Hölder inequality,

$$\begin{aligned}-\frac{\kappa^2}{\gamma}\int_{-\infty}^0 u_\varepsilon^3dx\le&\frac{\kappa^2}{|\gamma|}\int_{\mathbb{R}}|u_\varepsilon|^3dx\\\le&\frac{\kappa^2}{|\gamma|}\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^4(\mathbb{R})}\\\le&C(T)\|u_\varepsilon(t,\cdot)\|^2_{L^4(\mathbb{R})}\le C(T)\sqrt{\left(1+\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\right)}.\end{aligned}\tag{2.18}$$

(2.9) follows from (2.17) and (2.18).

We prove (2.10). By (2.3) and (2.12), we have

$$\int_0^\infty u_\varepsilon(t,x)dx=0.\tag{2.19}$$

Consequently, differentiating (2.19) with respect to t, we obtain

$$\frac{d}{dt}\int_0^\infty u_\varepsilon(t,x)dx=\int_0^\infty \partial_t u_\varepsilon(t,x)dx=0.\tag{2.20}$$

(2.3) and the the regularity of u_ε give

$$\lim_{x\to\infty}\left(+au_\varepsilon^3(t,x)-\varepsilon\partial_x u_\varepsilon(t,x)-bP_\varepsilon(t,x)\right)=0.\tag{2.21}$$

Therefore, by (2.15), (2.20) and (2.21),

$$\int_0^\infty P_\varepsilon(t,x)dx=-\frac{a}{\gamma}u_\varepsilon(t,0)+\frac{\varepsilon}{\gamma}\partial_x u_\varepsilon(t,0)-\frac{\kappa^2}{\gamma}\int_0^\infty u_\varepsilon^3(t,x)dx\tag{2.22}$$

Due to (2.6), (2.7) and the Hölder inequality,

$$\begin{aligned}-\frac{\kappa^2}{\gamma}\int_0^\infty u_\varepsilon^3dx\le&\frac{\kappa^2}{|\gamma|}\int_0^\infty|u_\varepsilon|^3dx\\\le&\frac{\kappa^2}{|\gamma|}\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^4(\mathbb{R})}\\\le&C(T)\|u_\varepsilon(t,\cdot)\|^2_{L^4(\mathbb{R})}\le C(T)\sqrt{\left(1+\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\right)}.\end{aligned}\tag{2.23}$$

(2.22) and (2.23) give (2.10).

Finally, we prove (2.11). Thanks to (2.9) and (2.10), we can consider (2.17) and (2.22) which give (2.11). □

Following [15, Lemma 2.4], we prove the following result.

Lemma 2.5. *Fix $T > 0$. There exists a constant $C(T) > 0$, independent on ε, such that*

$$\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{2.24}$$

In particular, for every $0 \le t \le T$, we have

$$\begin{aligned} \|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le& C(T),\\ \int_0^t \|P_\varepsilon(s,\cdot)u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds \le& C(T),\\ \|u_\varepsilon(t,\cdot)\|_{L^4(\mathbb{R})} \le& C(T),\\ \int_0^t \|u_\varepsilon(s,\cdot)\|^6_{L^6(\mathbb{R})} \le& C(T),\\ \varepsilon\int_0^t \|u_\varepsilon(s,\cdot)\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\,ds \le& C(T). \end{aligned} \tag{2.25}$$

Proof. Let $0 \le t \le T$. We begin by observing that, thanks to (2.9), we can consider the following function

$$F_\varepsilon(t,x) = \int_{-\infty}^x P_\varepsilon(t,y)dy. \tag{2.26}$$

Integrating the second equation of (2.1), by (2.3), we get

$$P_\varepsilon(t,x) = \int_{-\infty}^x u_\varepsilon(t,y)dy. \tag{2.27}$$

Differentiating (2.27) with respect to t, we obtain that

$$\partial_t P_\varepsilon(t,x) = \frac{d}{dt}\int_{-\infty}^x u_\varepsilon(t,y)dy = \int_{-\infty}^x \partial_t u_\varepsilon(t,y)dy. \tag{2.28}$$

Integrating the first equation of (2.1) on $(-\infty, x)$, from (2.26), (2.27) and (2.28), we have

$$\partial_t P_\varepsilon(t,x) = -au^3_\varepsilon(t,x) - bP_\varepsilon(t,x) - \kappa^2\int_{-\infty}^x u^3_\varepsilon(t,y)dy + \gamma F_\varepsilon(t,x) + \varepsilon\partial_x u_\varepsilon(t,x). \tag{2.29}$$

Multiplying (2.29) by $2P_\varepsilon$, an integration on $\mathbb{R}$ give

$$\begin{aligned} \frac{d}{dt}\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}} P_\varepsilon\partial_t P_\varepsilon dx\\ =& -2a\int_{\mathbb{R}} P_\varepsilon u^3_\varepsilon dx - 2b\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\int_{\mathbb{R}} P_\varepsilon\partial_x u_\varepsilon dx\\ &- 2\kappa^2\int_{\mathbb{R}} P_\varepsilon\left(\int_{-\infty}^x u^3_\varepsilon(t,y)dy\right)dx + 2\gamma\int_{\mathbb{R}} P_\varepsilon F_\varepsilon dx \end{aligned} \tag{2.30}$$

Observe that, by (2.11) and (2.26),

$$\begin{aligned} 2\gamma\int_{\mathbb{R}} P_\varepsilon F_\varepsilon dx =& 2\gamma\int_{\mathbb{R}} F_\varepsilon\partial_x F_\varepsilon dx = \gamma F^2_\varepsilon(t,\infty)\\ =& \gamma\left(\int_{\mathbb{R}} P_\varepsilon(t,x)dx\right)^2 = \frac{\kappa^4}{\gamma}\left(\int_{\mathbb{R}} u^3_\varepsilon dx\right)^2. \end{aligned} \tag{2.31}$$

Again by (2.11) and (2.26),

$$
\begin{aligned}
&-2\kappa^2\int_{\mathbb{R}}P_\varepsilon\left(\int_{-\infty}^{x}u_\varepsilon^3(t,y)dy\right)dx\\
&\quad=-2\kappa^2F_\varepsilon(t,\infty)\int_{\mathbb{R}}u_\varepsilon^3dx+2\kappa^2\int_{\mathbb{R}}F_\varepsilon u_\varepsilon^3dx\\
&\quad=-2\kappa^2\left(\int_{\mathbb{R}}P_\varepsilon dx\right)\left(\int_{\mathbb{R}}u_\varepsilon^3dx\right)+2\kappa^2\int_{\mathbb{R}}F_\varepsilon u_\varepsilon^3dx\\
&\quad=\frac{2\kappa^4}{\gamma}\left(\int_{\mathbb{R}}u_\varepsilon^3dx\right)^2+2\kappa^2\int_{\mathbb{R}}F_\varepsilon u_\varepsilon^3dx.
\end{aligned}\tag{2.32}
$$

Moreover, by (2.1) and (2.3),

$$
2\varepsilon\int_{\mathbb{R}}P_\varepsilon\partial_x u_\varepsilon dx=-2\varepsilon\int_{\mathbb{R}}\partial_xP_\varepsilon u_\varepsilon dx=-2\varepsilon\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\tag{2.33}
$$

Therefore, it follows from (2.30), (2.31), (2.32) and (2.33) that

$$
\begin{aligned}
&\frac{d}{dt}\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad=-2a\int_{\mathbb{R}}P_\varepsilon u_\varepsilon^3dx-2b\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad\quad+\frac{3\kappa^4}{\gamma}\left(\int_{\mathbb{R}}u_\varepsilon^3dx\right)^2+2\kappa^2\int_{\mathbb{R}}F_\varepsilon u_\varepsilon^3dx.
\end{aligned}\tag{2.34}
$$

By (2.29), we have

$$
F_\varepsilon(t,x)=\frac{1}{\gamma}\partial_tP_\varepsilon(t,x)+\frac{a}{\gamma}u_\varepsilon^3(t,x)+\frac{b}{\gamma}P_\varepsilon(t,x)+\frac{\kappa^2}{\gamma}\int_{-\infty}^{x}u_\varepsilon^3(t,y)dy-\frac{\varepsilon}{\gamma}\partial_xu_\varepsilon(t,x).\tag{2.35}
$$

Multiplying (2.35) by $2\kappa^2u_\varepsilon^3$, an integration on $\mathbb{R}$ gives

$$
\begin{aligned}
2\kappa^2\int_{\mathbb{R}}F_\varepsilon u_\varepsilon^3dx=&\frac{2\kappa^2}{\gamma}\int_{\mathbb{R}}\partial_tP_\varepsilon u_\varepsilon^3dx+\frac{2a\kappa^2}{\gamma}\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}+\frac{2\kappa^2b}{\gamma}\int_{\mathbb{R}}P_\varepsilon u_\varepsilon^3dx\\
&+\frac{2\kappa^4}{\gamma}\int_{\mathbb{R}}u_\varepsilon^3\left(\int_{-\infty}^{x}u_\varepsilon^3dy\right)dx-\frac{2\kappa^2\varepsilon}{\gamma}\int_{\mathbb{R}}u_\varepsilon^3\partial_xu_\varepsilon dx.
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{2\kappa^4}{\gamma}\int_{\mathbb{R}}u_\varepsilon^3\left(\int_{-\infty}^{x}u_\varepsilon^3dy\right)dx=&\frac{\kappa^4}{\gamma}\left(\int_{\mathbb{R}}u_\varepsilon^3dx\right)^2,\\
-\frac{2\kappa^2\varepsilon}{\gamma}\int_{\mathbb{R}}u_\varepsilon^3\partial_xu_\varepsilon dx=&0,
\end{aligned}
$$

we have that

$$
\begin{aligned}
2\kappa^2\int_{\mathbb{R}}F_\varepsilon u_\varepsilon^3dx=&\frac{2\kappa^2}{\gamma}\int_{\mathbb{R}}\partial_tP_\varepsilon u_\varepsilon^3dx+\frac{2a\kappa^2}{\gamma}\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\
&+\frac{2\kappa^2b}{\gamma}\int_{\mathbb{R}}P_\varepsilon u_\varepsilon^3dx+\frac{\kappa^4}{\gamma}\left(\int_{\mathbb{R}}u_\varepsilon^3dx\right)^2.
\end{aligned}\tag{2.36}
$$

Using (2.36) in (2.34), we get

$$
\frac{d}{dt}\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}
$$

$$
\begin{aligned}
&= \frac{2(\kappa^2 b - a\gamma)}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx - 2b\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + \frac{2a\kappa^2}{\gamma}\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})} + \frac{4\kappa^4}{\gamma}\left(\int_{\mathbb{R}} u_\varepsilon^3 dx\right)^2 + \frac{2\kappa^2}{\gamma}\int_{\mathbb{R}} \partial_t P_\varepsilon u_\varepsilon^3 dx.
\end{aligned} \tag{2.37}
$$

Observe that

$$\partial_t(P_\varepsilon u_\varepsilon^3) = \partial_t P_\varepsilon u_\varepsilon^3 + P_\varepsilon \partial_t u_\varepsilon^3 = \partial_t P_\varepsilon u_\varepsilon^3 + 3P_\varepsilon u_\varepsilon^2 \partial_t u_\varepsilon.$$

Consequently,

$$\frac{2\kappa^2}{\gamma}\int_{\mathbb{R}} \partial_t P_\varepsilon u_\varepsilon^3 dx = \frac{2\kappa^2}{\gamma}\frac{d}{dt}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx - \frac{6\kappa^2}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^2 \partial_t u_\varepsilon dx dx. \tag{2.38}$$

Using (2.38) in (2.37), we have

$$
\begin{aligned}
\frac{dG(t)}{dt} + 2\varepsilon\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &= \frac{2(\kappa^2 b - a\gamma)}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx\\
&\quad - 2b\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{2a\kappa^2}{\gamma}\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\
&\quad + \frac{4\kappa^4}{\gamma}\left(\int_{\mathbb{R}} u_\varepsilon^3 dx\right)^2 - \frac{6\kappa^2}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^2 \partial_t u_\varepsilon dx,
\end{aligned} \tag{2.39}
$$

where

$$G(t) := \left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{2\kappa^2}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx. \tag{2.40}$$

Multiplying the first equation of (2.1) by $-\frac{6\kappa^2}{\gamma}P_\varepsilon u_\varepsilon^2$, an integration on $\mathbb{R}$ give

$$
\begin{aligned}
-\frac{6\kappa^2}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^2 \partial_t u_\varepsilon dx &= \frac{18a\kappa^2}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^4 \partial_x u_\varepsilon dx + \frac{6b\kappa^2}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx\\
&\quad + \frac{\kappa^4}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^5 dx - 6\kappa^2\left\|P_\varepsilon(t,\cdot)u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad - \frac{6\kappa^2\varepsilon}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^2 \partial_{xx}^2 u_\varepsilon dx.
\end{aligned} \tag{2.41}
$$

Observe that by (2.1) and (2.3),

$$
\begin{aligned}
\frac{18a\kappa^2}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^4 \partial_x u_\varepsilon dx &= -\frac{18a\kappa^2}{5\gamma}\int_{\mathbb{R}} u_\varepsilon^5 \partial_x P_\varepsilon dx = -\frac{18a\kappa^2}{5\gamma}\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})},\\
-\frac{6\kappa^2\varepsilon}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^2 \partial_{xx}^2 u_\varepsilon dx &= \frac{6\kappa^2\varepsilon}{\gamma}\int_{\mathbb{R}} \partial_x P_\varepsilon u_\varepsilon^2 \partial_x u_\varepsilon dx + \frac{12\kappa^2\varepsilon}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon(\partial_x u_\varepsilon)^2 dx\\
&= \frac{6\kappa^2\varepsilon}{\gamma}\int_{\mathbb{R}} u_\varepsilon^3 \partial_x u_\varepsilon dx + \frac{12\kappa^2\varepsilon}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon(\partial_x u_\varepsilon)^2 dx\\
&= \frac{12\kappa^2\varepsilon}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon(\partial_x u_\varepsilon)^2 dx.
\end{aligned}
$$

Hence, by (2.41),

$$\begin{aligned}
-\frac{6\kappa^2}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^2 \partial_t u_\varepsilon dx =& -\frac{18a\kappa^2}{5\gamma}\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})} + \frac{6b\kappa^2}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx\\
&+\frac{\kappa^4}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^5 dx - 6\kappa^2\left\|P_\varepsilon(t,\cdot)u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&+\frac{12\kappa^2\varepsilon}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon(\partial_x u_\varepsilon)^2 dx.
\end{aligned}\tag{2.42}$$

Substituting (2.42) in (2.39), we get

$$\begin{aligned}
&\frac{dG(t)}{dt} + 2\varepsilon\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 6\kappa^2\left\|P_\varepsilon(t,\cdot)u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad= \frac{8\kappa^2 b - 2a\gamma}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx - 2b\left\|P_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{8a\kappa^2}{\gamma}\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\
&\quad+\frac{4\kappa^4}{\gamma}\left(\int_{\mathbb{R}} u_\varepsilon^3 dx\right)^2 + \frac{\kappa^4}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^5 dx + \frac{12\kappa^2\varepsilon}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon(\partial_x u_\varepsilon)^2 dx.
\end{aligned}\tag{2.43}$$

Due to (2.6), (2.7) and the Young inequality and the Hölder inequality,

$$\begin{aligned}
&\frac{|8\kappa^2 b - 2a\gamma|}{|\gamma|}\int_{\mathbb{R}} |P_\varepsilon||u_\varepsilon|^3 dx = \frac{|8\kappa^2 b - 2a\gamma|}{|\gamma|}\int_{\mathbb{R}} |P_\varepsilon u_\varepsilon| u_\varepsilon^2 dx\\
&\quad\le \frac{|8\kappa^2 b - 2a\gamma|}{2|\gamma|}\int_{\mathbb{R}} P_\varepsilon^2 u_\varepsilon^2 dx + \frac{|8\kappa^2 b - 2a\gamma|}{2|\gamma|}\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^4(\mathbb{R})}\\
&\quad\le \frac{|8\kappa^2 b - 2a\gamma|}{2|\gamma|}\left\|P_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left(1 + \left\|P_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}\right)\\
&\quad\le C(T)\left\|P_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + C(T),\\
&\frac{\kappa^4}{|\gamma|}\int_{\mathbb{R}} |P_\varepsilon|u_\varepsilon|^5 dx = \frac{\kappa^4}{|\gamma|}\int_{\mathbb{R}} |\sqrt{D}P_\varepsilon u_\varepsilon^2|\frac{u_\varepsilon^3}{\sqrt{D}} dx\\
&\quad\le \frac{D\kappa^4}{2|\gamma|}\int_{\mathbb{R}} P_\varepsilon^2 u_\varepsilon^4 dx + \frac{\kappa^4}{2D|\gamma|}\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\
&\quad= \int_{\mathbb{R}}\left|\frac{D\kappa^4 P_\varepsilon^2 u_\varepsilon^2}{2\sqrt{E}|\gamma|}\right|\left|\sqrt{E}u_\varepsilon^3\right| dx + \frac{\kappa^4}{2D|\gamma|}\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\
&\quad\le \frac{D^2\kappa^8}{4E\gamma^2}\int_{\mathbb{R}} P_\varepsilon^4 u_\varepsilon^2 dx + \left(\frac{E}{2} + \frac{\kappa^4}{2D|\gamma|}\right)\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\
&\quad\le \frac{D^2\kappa^8}{4E\gamma^2}\left\|P_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbb{R})}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left(\frac{E}{2} + \frac{\kappa^4}{2D|\gamma|}\right)\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})}\\
&\quad\le \frac{D^2 C(T)}{E}\left\|P_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbb{R})} + \left(\frac{E}{2} + \frac{\kappa^4}{2D|\gamma|}\right)\left\|u_\varepsilon(t,\cdot)\right\|^6_{L^6(\mathbb{R})},\\
&\frac{12\kappa^2\varepsilon}{|\gamma|}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon(\partial_x u_\varepsilon)^2 dx = \frac{12\kappa^2\varepsilon}{|\gamma|}\int_{\mathbb{R}} |P_\varepsilon\partial_x u_\varepsilon||u_\varepsilon\partial_x u_\varepsilon| dx\\
&\quad\le \frac{6\kappa^2\varepsilon}{|\gamma|}\int_{\mathbb{R}} P_\varepsilon^2(\partial_x u_\varepsilon)^2 dx + \frac{6\kappa^2\varepsilon}{|\gamma|}\left\|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le \frac{6\kappa^2\varepsilon}{|\gamma|}\left\|P_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{6\kappa^2\varepsilon}{|\gamma|}\left\|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

$$\begin{aligned}\frac{4\kappa^4}{|\gamma|}\left(\int_{\mathbb{R}}|u_\varepsilon|^3dx\right)^2&\le\frac{4\kappa^4}{|\gamma|}\left(\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^4_{L^2(\mathbb{R})}\right)^2\\&\le C(T)\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})}\le C(T)\left(1+\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\right)\\&\le C(T)+C(T)\|P_\varepsilon(t,\cdot)\|^2_{L^\infty((0,T)\times\mathbb{R})},\end{aligned}$$

where D, E are two positive constants which will be specified later. Consequently, by (2.43),

$$\begin{aligned}&\frac{dG(t)}{dt}+2\varepsilon\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+6\kappa^2\|P_\varepsilon(t,\cdot)u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\quad\le 2|b|\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\left(\frac{8|a|\kappa^2}{|\gamma|}+\frac{E}{2}+\frac{\kappa^4}{2D|\gamma|}\right)\|u_\varepsilon(t,\cdot)\|^6_{L^6(\mathbb{R})}\\&\qquad+\frac{6\kappa^2\varepsilon}{|\gamma|}\|P_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{6\kappa^2\varepsilon}{|\gamma|}\|u_\varepsilon(t,\cdot)\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\qquad+\frac{D^2C(T)}{E}\|P_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}+C(T)\|P_\varepsilon(t,\cdot)\|^2_{L^\infty((0,T)\times\mathbb{R})}+C(T).\end{aligned}\tag{2.44}$$

Observe that by (2.40),

$$2|b|\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}=2|b|G(t)+\frac{4|b|\kappa^2}{\gamma}\int_{\mathbb{R}}P_\varepsilon u_\varepsilon^3dx.\tag{2.45}$$

Thanks to (2.6), (2.7) and the Young inequality,

$$\begin{aligned}\frac{4|b|\kappa^2}{|\gamma|}\int_{\mathbb{R}}|P_\varepsilon||u_\varepsilon|^3dx&=\frac{4|b|\kappa^2}{|\gamma|}\int_{\mathbb{R}}|P_\varepsilon u_\varepsilon|u_\varepsilon^2dx\\&\le\frac{2|b|\kappa^2}{|\gamma|}\int_{\mathbb{R}}P_\varepsilon^2u_\varepsilon^2dx+\frac{2|b|\kappa^2}{|\gamma|}\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})}\\&\le\frac{2|b|\kappa^2}{|\gamma|}\|P_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(T)\left(1+\|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\right)\\&\le C(T)\|P_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}+C(T).\end{aligned}\tag{2.46}$$

It follows from (2.44), (2.45) and (2.46) that

$$\begin{aligned}&\frac{dG(t)}{dt}+2\varepsilon\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+6\kappa^2\|P_\varepsilon(t,\cdot)u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\quad\le 2|b|G(t)+\left(\frac{8|a|\kappa^2}{|\gamma|}+\frac{E}{2}+\frac{\kappa^4}{2D|\gamma|}\right)\|u_\varepsilon(t,\cdot)\|^6_{L^6(\mathbb{R})}\\&\qquad+\frac{6\kappa^2\varepsilon}{|\gamma|}\|P_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{6\kappa^2\varepsilon}{|\gamma|}\|u_\varepsilon(t,\cdot)\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\&\qquad+\frac{D^2C(T)}{E}\|P_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}+C(T)\|P_\varepsilon(t,\cdot)\|^2_{L^\infty((0,T)\times\mathbb{R})}+C(T).\end{aligned}$$

The Gronwall Lemma, (2.2), (2.6), (2.7), (2.40), (2.46) and the Young inequality give

$$
\begin{aligned}
&\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon e^{2|b|t}\int_0^t e^{-2|b|s}\|u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\quad + 6\kappa^2 e^{2|b|t}\int_0^t e^{-2|b|s}\|P_\varepsilon(s,\cdot)u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\le C_0 + \frac{2\kappa^2}{\gamma}\int_{\mathbb{R}} P_\varepsilon u_\varepsilon^3 dx\\
&\quad + \left(\frac{8|a|\kappa^2}{|\gamma|} + \frac{E}{2} + \frac{\kappa^4}{2D|\gamma|}\right)e^{2|b|t}\int_0^t e^{-2|b|s}\|u_\varepsilon(s,\cdot)\|^6_{L^6(\mathbb{R})}ds\\
&\quad + \frac{6\kappa^2\varepsilon}{|\gamma|}\|P_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}e^{2|b|t}\int_0^t e^{-2|b|s}\|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\quad + \frac{6\kappa^2\varepsilon}{|\gamma|}e^{2|b|t}\int_0^t e^{-2|b|s}\|u_\varepsilon(s,\cdot)\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\
&\quad + \frac{D^2C(T)}{E}\|P_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}e^{2|b|t}\int_0^t e^{-2|b|s}ds\\
&\quad + C(T)\|P_\varepsilon(t,\cdot)\|^2_{L^\infty((0,T)\times\mathbb{R})}e^{2|b|t}\int_0^t e^{-2|b|s}ds + C(T)e^{2|b|t}\int_0^t e^{-2|b|s}ds\\
&\le C(T)\left(\frac{8|a|\kappa^2}{|\gamma|} + \frac{E}{2} + \frac{\kappa^4}{2D|\gamma|}\right)\int_0^t \|u_\varepsilon(s,\cdot)\|^6_{L^6(\mathbb{R})}ds\\
&\quad + C(T)\varepsilon\int_0^t \|u_\varepsilon(s,\cdot)\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}ds + \frac{D^2C(T)}{E}\|P_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}\\
&\quad \le C(T)\|P_\varepsilon(t,\cdot)\|^2_{L^\infty((0,T)\times\mathbb{R})} + C(T)\\
&\le C(T)\left(\frac{E}{2} + \frac{\kappa^4}{2D|\gamma|}\right)\left(1 + \|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\right)\\
&\quad + C(T)\left(1 + \|P_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\right)\\
&\quad + \frac{D^2C(T)}{E}\|P_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} + C(T)\|P_\varepsilon(t,\cdot)\|^2_{L^\infty((0,T)\times\mathbb{R})} + C(T)\\
&\le C(T)\left(\frac{E}{2} + \frac{\kappa^4}{2D|\gamma|} + 1\right)\left(1 + \|P_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\\
&\quad + \frac{D^2C(T)}{E}\|P_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} + C(T)\|P_\varepsilon(t,\cdot)\|^2_{L^\infty((0,T)\times\mathbb{R})} + C(T).
\end{aligned}
\tag{2.47}
$$

We prove (2.24). Thanks to (2.1), (2.3), (2.6) and the Hölder inequality,

$$
\begin{aligned}
P_\varepsilon^2(t,x) =&2\int_{-\infty}^x P_\varepsilon\partial_x P_\varepsilon dy \le 2\int_{\mathbb{R}}|P_\varepsilon||u_\varepsilon|dx\\
&\le 2\|P_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le C(T)\|P_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, by (2.47),

$$
\begin{aligned}
\|P_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} \le& C(T)\|P_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
\le& C(T)\left(\frac{E}{2} + \frac{\kappa^4}{2D|\gamma|} + 1\right)\left(1 + \|P_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\\
&+ \frac{D^2C(T)}{E}\|P_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} + C(T)\|P_\varepsilon(t,\cdot)\|^2_{L^\infty((0,T)\times\mathbb{R})} + C(T).
\end{aligned}
$$

Hence,

$$\left(1-\frac{D^2C(T)}{E}\right)\|P_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}-C(T)\left(\frac{E}{2}+\frac{\kappa^4}{2D|\gamma|}+1\right)\left(1+\|P_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)$$
$$-C(T)\|P_\varepsilon(t,\cdot)\|^2_{L^\infty((0,T)\times\mathbb{R})}-C(T)\leq 0.$$

Choosing

$$E=D,\quad D=\frac{1}{2C(T)}, \tag{2.48}$$

we have

$$\frac{1}{2}\|P_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}-C(T)\|P_\varepsilon(t,\cdot)\|^2_{L^\infty((0,T)\times\mathbb{R})}-C(T)\leq 0,$$

which gives (2.24).

Finally (2.25) follows from (2.7), (2.24), (2.47) and (2.48). □

Following [9, Lemma 3.1], or [20, Lemma 3.1], we prove the following result.

Lemma 2.6. *Let $T>0$. There exists a constant $C(T)>0$, independent on ε, such that*

$$\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\leq C(T), \tag{2.49}$$

for every $0\leq t\leq T$.

Proof. Let $0\leq t\leq T$. We begin by observing that, by (2.24), we have that

$$|\gamma P_\varepsilon(t,x)|\leq|\gamma|C(T),\quad (t,x)\in(0,T)\times\mathbb{R}. \tag{2.50}$$

Therefore,

$$-|\gamma|C(T)\leq\gamma P_\varepsilon(t,x)\leq|\gamma|C(T). \tag{2.51}$$

The proof of (2.49) splits into two parts. In the first part, we consider $b\geq 0$. Instead, in the second one, we consider $b\leq 0$.

Case $b\geq 0$. We assume that

$$b=\alpha^2. \tag{2.52}$$

Therefore, by the first equation of (2.1), (2.51) and (2.52), we have

$$\partial_t u_\varepsilon+a\partial_x u_\varepsilon^3-\varepsilon\partial^2_{xx}u_\varepsilon\leq|\gamma|C(T)-\alpha^2u_\varepsilon-\kappa^2u_\varepsilon^3. \tag{2.53}$$

A supersolution of (2.1) satisfies the following ordinary differential equation:

$$\frac{dz_1}{dt}+\alpha^2z_1+\kappa^2z_1^3-|\gamma|C(T)=0,\quad z_1(0)=\|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})}. \tag{2.54}$$

We consider the map

$$Z_1(t)=At+A,\quad t\geq 0. \tag{2.55}$$

where A is a positive constant, which will be specified later. Observe that

$$\frac{dZ_1}{dt}+\alpha^2Z_1+\kappa^2Z_1^3-|\gamma|C(T)=A+\alpha^2A(t+1)+\kappa^2A(t+1)^3-|\gamma|C(T)$$

Choosing

$$A=|\gamma|C(T), \tag{2.56}$$

we have that

$$\frac{dZ_1}{dt}+\alpha^2Z_1+\kappa^2Z_1^3-|\gamma|C(T)=\alpha^2|\gamma|C(T)(t+1)+\kappa^2|\gamma|C(T)(t+1)^3\geq 0, \tag{2.57}$$

for every $t \in (0,T)$. Then, $Z_1(t)$ is a supersolution of (2.54). (2.56), the comparison principle for parabolic equations and the comparison principle for ordinary differential equations yield

$$u_\varepsilon(t,x) \le z_1(t) \le Z_1(t) = |\gamma| C(T)(t+1), \quad (t,x) \in (0,T)\times\mathbb{R}. \tag{2.58}$$

Observe that, by the first equation of (2.1), (2.51) and (2.52), we have

$$\partial_t u_\varepsilon + a\partial_x u_\varepsilon^3 - \varepsilon\partial_{xx}^2 u_\varepsilon \ge -|\gamma| C(T) - \alpha^2 u_\varepsilon - \kappa^2 u_\varepsilon^3. \tag{2.59}$$

Therefore, a subsolution of (2.1) satisfies the following ordinary differential equation:

$$\frac{dz_2}{dt} + \alpha^2 z_2 + \kappa^2 z_2^3 + |\gamma| C(T) = 0, \quad z_2(0) = \|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})}. \tag{2.60}$$

We consider the map

$$Z_2(t) = -Bt - B, \quad t \ge 0. \tag{2.61}$$

where B is a positive constant, which will be specified later. Observe that

$$\frac{dZ_2}{dt} + \alpha^2 Z_2 + \kappa^2 Z_2^3 + |\gamma| C(T) = -B - \alpha^2 B(t+1) - \kappa^2 B(t+1)^3 + |\gamma| C(T).$$

Choosing

$$B = |\gamma| C(T), \tag{2.62}$$

we have that

$$\frac{dZ_1}{dt} + \alpha^2 Z_1 + \kappa^2 Z_1^3 - |\gamma| C(T) = -\alpha^2|\gamma| C(T)(t+1) - \kappa^2|\gamma| C(T)(t+1)^3 \le 0, \tag{2.63}$$

for every $t \in (0,T)$. Then, $Z_2(t)$ is a subsolution of (2.61). (2.62), the comparison principle for parabolic equations and the comparison principle for ordinary differential equations yield

$$-|\gamma| C(T)(t+1) = Z_2(t) \le z_2(t) \le u_\varepsilon(t,x), \quad (t,x) \in (0,T)\times\mathbb{R}. \tag{2.64}$$

It follows from (2.58) and (2.64) that

$$|u_\varepsilon(t,x)| \le |\gamma| C(T)(t+1) \le |\gamma| C(T)(T+1), \tag{2.65}$$

which give (2.49).

Case $b \le 0$. We assume that

$$b = -\alpha^2. \tag{2.66}$$

Thanks to (2.66), arguing as in previous case, we get

$$\partial_t u_\varepsilon + a\partial_x u_\varepsilon^3 - \varepsilon\partial_{xx}^2 u_\varepsilon \le |\gamma| C(T) + \alpha^2 u_\varepsilon - \kappa^2 u_\varepsilon^3. \tag{2.67}$$

A supersolution of (2.1) satisfies the following ordinary differential equation:

$$\frac{dz_3}{dt} - \alpha^2 z_3 + \kappa^2 z_3^3 - |\gamma| C(T) = 0, \quad z_3(0) = \|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})}. \tag{2.68}$$

We consider the map

$$Z_3(t) = Dt + E, \quad t \ge 0. \tag{2.69}$$

where D, E are two positive constants, which will be specified later. Observe that

$$\begin{aligned} &\frac{dZ_3}{dt} - \alpha^2 Z_3 + \kappa^2 Z_3^3 - |\gamma| C(T) \\ =&D - \alpha^2 (Dt+E) + \kappa^2 (Dt+E)^3 + |\gamma| C(T) \\ =&\kappa^2 D^3 t^3 + 3\kappa^2 D^2 E t^2 + D\left(3\kappa^2 E^2 - \alpha^2\right)t + D + \kappa^2 E^3 - \alpha^2 E - |\gamma| C(T). \end{aligned} \tag{2.70}$$

We search D, E such that,

$$3\kappa^2E^2-\alpha^2\geq 0,\quad D+\kappa^2E^3-\alpha^2E-|\gamma|C(T)\geq 0. \tag{2.71}$$

From the first inequality of (2.71), we obtain that

$$E\geq\left|\frac{\alpha}{\sqrt{3}\kappa}\right|. \tag{2.72}$$

Choosing

$$D=|\gamma|C(T), \tag{2.73}$$

it follows from the second inequality of (2.71) that

$$\kappa^2E^3-\alpha^2E\geq 0\quad\Rightarrow\quad\kappa^2E^2-\alpha^2\geq 0,$$

that is

$$E\geq\left|\frac{\alpha}{\kappa}\right|. \tag{2.74}$$

From (2.72) and (2.74), we get

$$E\geq\max\left\{\left|\frac{\alpha}{\sqrt{3}\kappa}\right|,\left|\frac{\alpha}{\kappa}\right|\right\}=\left|\frac{\alpha}{\kappa}\right|. \tag{2.75}$$

Choosing

$$E=\left|\frac{\alpha}{\kappa}\right|, \tag{2.76}$$

from (2.69) and (2.73), we have that

$$Z_3(t)=Dt+E=|\gamma|C(T)t+\left|\frac{\alpha}{\kappa}\right|. \tag{2.77}$$

Moreover, by (2.70), (2.73) and (2.76),

$$\frac{dZ_3}{dt}-\alpha^2Z_3+\kappa^2Z_3^3-|\gamma|C(T)\geq 0,$$

for every $0\leq t\leq T$. Then, $Z_3(t)$ is a supersolution of (2.68). (2.77), the comparison principle for parabolic equations and the comparison principle for ordinary differential equations yield

$$u_\varepsilon(t,x)\leq z_3(t)\leq Z_3(t)=|\gamma|C(T)t+\left|\frac{\alpha}{\kappa}\right|,\quad (t,x)\in(0,T)\times\mathbb{R}. \tag{2.78}$$

Arguing as in previous case, we have that

$$\partial_tu_\varepsilon+a\partial_xu_\varepsilon^3-\varepsilon\partial_{xx}^2u_\varepsilon\geq-|\gamma|C(T)+\alpha^2u_\varepsilon-\kappa^2u_\varepsilon^3. \tag{2.79}$$

Therefore, a subsolution of (2.1) satisfies the following ordinary differential equation:

$$\frac{dz_4}{dt}-\alpha^2z_4+\kappa^2z_4^3+|\gamma|C(T)=0,\quad z_2(0)=\|u_{\varepsilon,0}\|_{L^\infty(\mathbb{R})}. \tag{2.80}$$

We consider the map

$$Z_4(t)=-Ft-G,\quad t\geq 0. \tag{2.81}$$

where F, G are two positive constants, which will be specified later. Observe that

$$\begin{aligned}&\frac{dZ_4}{dt}-\alpha^2Z_4+\kappa^2Z_4^3+|\gamma|C(T)\\&=-F+\alpha^2(Ft+G)-\kappa^2(Ft+G)^3+|\gamma|C(T)\\&=-\kappa^2F^3t^3-3\kappa^2F^2Gt^2+F\left(\alpha^2-3\kappa^2G^2\right)t-F+\alpha^2G-\kappa^2G^3+|\gamma|C(T)\end{aligned} \tag{2.82}$$

We search F, G such that

$$\alpha^2 - 3\kappa^2 G^2 \le 0, \quad -F + \alpha^2 G - \kappa^2 G^3 + |\gamma| C(T) \le 0. \tag{2.83}$$

Choosing

$$F = |\gamma| C(T), \tag{2.84}$$

by (2.83), we have

$$3\kappa^2 G^2 - \alpha^2 \ge 0, \quad \kappa^2 G^3 - \alpha^2 G \ge 0.$$

Arguing as before, we gain

$$G \ge \max\left\{ \left|\frac{\alpha}{\sqrt{3}\kappa}\right|, \left|\frac{\alpha}{\kappa}\right| \right\} = \left|\frac{\alpha}{\kappa}\right|. \tag{2.85}$$

Choosing

$$G = \left|\frac{\alpha}{\kappa}\right|, \tag{2.86}$$

then, by (2.81) and (2.84),

$$Z_4(t) = -Ft - G = -|\gamma| C(T) t - \left|\frac{\alpha}{\kappa}\right|. \tag{2.87}$$

Moreover, by (2.82), (2.84) and (2.86), we have

$$\frac{dZ_4}{dt} - \alpha^2 Z_4 + \kappa^2 Z_4^3 + |\gamma| C(T) \le 0,$$

for every $0 \le t \le T$. Then, $Z_4(t)$ is a subsolution of (2.80). (2.87), the comparison principle for parabolic equations and the comparison principle for ordinary differential equations yield

$$-|\gamma| C(T) t - \left|\frac{\alpha}{\kappa}\right| \le Z_4(t) \le z_4(t) \le u_\varepsilon(t,x), \quad (t,x) \in (0,T) \times \mathbb{R}. \tag{2.88}$$

It follows from (2.78) and (2.88) that

$$-|\gamma| C(T) t - \left|\frac{\alpha}{\kappa}\right| \le u_\varepsilon(t,x) \le |\gamma| C(T) t + \left|\frac{\alpha}{\kappa}\right|.$$

Hence,

$$|u_\varepsilon(t,x)| \le |\gamma| C(T) t + \left|\frac{\alpha}{\kappa}\right| \le |\gamma| C(T) T + \left|\frac{\alpha}{\kappa}\right|,$$

which gives (2.49). □

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Let us begin by proving the existence of a distributional solution to (1.10) satisfying (1.13).

Lemma 3.1. *Let $T > 0$. There exists a function $u \in L^\infty((0,T)\times\mathbb{R})$ that is a distributional solution of* (2.1) *and satisfies* (1.13) *for every convex entropy $\eta \in C^2(\mathbb{R})$.*

We construct a solution by passing to the limit in a sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of viscosity approximations (2.1). We use the compensated compactness method [61].

Lemma 3.2. *Let $T > 0$. There exists a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and a limit function $u \in L^\infty((0,T)\times\mathbb{R})$ such that*

$$u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{loc}((0,T)\times\mathbb{R}),\ 1 \le p < \infty. \tag{3.1}$$

Moreover, we have that

$$P_{\varepsilon_k} \to P \text{ a.e. and in } L^p_{loc}((0,T); W^{1,p}_{loc}(\mathbb{R})),\ 1 \le p < \infty, \tag{3.2}$$

where

$$P(t,x)=\int_0^x u(t,y)dy, \qquad t\ge 0, \quad x\in\mathbb{R}. \tag{3.3}$$

Proof. Let $\eta:\mathbb{R}\to\mathbb{R}$ be any convex C^2 entropy function, and $q:\mathbb{R}\to\mathbb{R}$ be the corresponding entropy flux defined by $q'(u)=3au^2\eta'(u)$. By multiplying the first equation in (2.1) with $\eta'(u_\varepsilon)$ and using the chain rule, we get

$$\partial_t\eta(u_\varepsilon)+\partial_x q(u_\varepsilon)=\underbrace{\varepsilon\partial_{xx}^2\eta(u_\varepsilon)}_{=:\mathcal{L}_{1,\varepsilon}}\underbrace{-\varepsilon\eta''(u_\varepsilon)\left(\partial_x u_\varepsilon\right)^2}_{=:\mathcal{L}_{2,\varepsilon}}\underbrace{+\gamma\eta'(u_\varepsilon)P_\varepsilon}_{=:\mathcal{L}_{3,\varepsilon}}\underbrace{-b\eta'(u_\varepsilon)u_\varepsilon}_{=:\mathcal{L}_{4,\varepsilon}}\underbrace{-\kappa^2\eta'(u_\varepsilon)u_\varepsilon^3}_{=:\mathcal{L}_{5,\varepsilon}}$$

where $\mathcal{L}_{1,\varepsilon}$, $\mathcal{L}_{2,\varepsilon}$, $\mathcal{L}_{3,\varepsilon}$, $\mathcal{L}_{4,\varepsilon}$ and $\mathcal{L}_{5,\varepsilon}$ are distributions. Let us show that

$$\mathcal{L}_{1,\varepsilon}\to 0 \text{ in } H^{-1}((0,T)\times\mathbb{R}),\ T>0.$$

Since

$$\varepsilon\partial_{xx}^2\eta(u_\varepsilon)=\partial_x(\varepsilon\eta'(u_\varepsilon)\partial_x u_\varepsilon),$$

by (2.6) and Lemma 2.6,

$$\begin{aligned}\left\|\varepsilon\eta'(u_\varepsilon)\partial_x u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})}&\le\varepsilon^2\left\|\eta'\right\|^2_{L^\infty(-C(T),C(T))}\int_0^T\left\|\partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\&\le\varepsilon\left\|\eta'\right\|^2_{L^\infty(-C(T),C(T))}C(T)\to 0.\end{aligned}$$

We claim that

$$\{\mathcal{L}_{2,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1((0,T)\times\mathbb{R}),\ T>0.$$

Again by (2.6) and Lemma 2.6,

$$\begin{aligned}\left\|\varepsilon\eta''(u_\varepsilon)(\partial_x u_\varepsilon)^2\right\|_{L^1((0,T)\times\mathbb{R})}&\le\left\|\eta''\right\|_{L^\infty(-C(T),C(T))}\varepsilon\int_0^T\left\|\partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\&\le\left\|\eta''\right\|_{L^\infty(-C(T),C(T))}C(T).\end{aligned}$$

We have that

$$\{\mathcal{L}_{3,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1_{loc}((0,T)\times\mathbb{R}),\ T>0.$$

Let K be a compact subset of $(0,T)\times\mathbb{R}$. Using (2.24) and Lemma 2.6,

$$\begin{aligned}|\gamma|\left\|\eta'(u_\varepsilon)P_\varepsilon\right\|_{L^1(K)}&=|\gamma|\iint_K|\eta'(u_\varepsilon)||P_\varepsilon|dtdx\\&\le|\gamma|\left\|\eta'\right\|_{L^\infty(-C(T),C(T))}\left\|P_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}|K|.\end{aligned}$$

We show

$$\{\mathcal{L}_{4,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1_{loc}((0,T)\times\mathbb{R}),\ T>0.$$

Let K be a compact subset of $(0,T)\times\mathbb{R}$. By Lemma 2.6,

$$\begin{aligned}|b|\left\|\eta'(u_\varepsilon)u_\varepsilon\right\|_{L^1(K)}&=|b|\iint_K|\eta'(u_\varepsilon)||u_\varepsilon|dtdx\\&\le|b|\left\|\eta'\right\|_{L^\infty(-C(T),C(T))}\left\|u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}|K|\\&\le\left\|\eta'\right\|_{L^\infty(-C(T),C(T))}|K|C(T).\end{aligned}$$

We claim that

$$\{\mathcal{L}_{5,\varepsilon}\}_{\varepsilon>0} \text{ is uniformly bounded in } L^1_{loc}((0,T)\times\mathbb{R}),\ T>0.$$

Let K be a compact subset of $(0,T)\times\mathbb{R}$. Again by Lemma 2.6,

$$\begin{aligned}\kappa^2\left\|\eta'(u_\varepsilon)u_\varepsilon^3\right\|_{L^1(K)} &= \kappa^2\iint_K |\eta'(u_\varepsilon)||u_\varepsilon|^3 dtdx\\ &\le \kappa^2\left\|\eta'\right\|_{L^\infty(-C(T),C(T))}\left\|u_\varepsilon\right\|^3_{L^\infty((0,T)\times\mathbb{R})}|K|\\ &\le \left\|\eta'\right\|_{L^\infty(-C(T),C(T))}|K|C(T).\end{aligned}$$

Therefore, Murat's lemma [47] implies that

$$\{\partial_t\eta(u_\varepsilon)+\partial_x q(u_\varepsilon)\}_{\varepsilon>0} \text{ lies in a compact subset of } H^{-1}_{\mathrm{loc}}((0,T)\times\mathbb{R}). \tag{3.4}$$

The L^∞ bound stated in Lemma 2.6, (3.4) and the Tartar's compensated compactness method [61] give the existence of a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ and a limit function $u\in L^\infty((0,T)\times\mathbb{R})$, $T>0$, such that (3.1) holds.

Finally, (3.2) follows from (3.1), the Hölder inequality and the identity

$$P_{\varepsilon_k}=\int_0^x u_{\varepsilon_k}dy,\quad \partial_x P_{\varepsilon_k}=u_{\varepsilon_k}.$$

□

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Lemma 3.2 gives the existence of an entropy solution u for (1.9), or equivalently (1.10).

We prove that $u(t,x)$ is unique and (1.14) holds. Fix $T>0$. Let $u(t,x)$ and $v(t,x)$ be two entropy solution of (1.9), or (1.10) such that

$$u,\, v\in L^\infty((0,T)\times\mathbb{R}) \tag{3.5}$$

Consequently, by (3.5), we have that

$$\left|u^3-v^3\right|\le C(T)|u-v|, \tag{3.6}$$

where

$$C(T)=\frac{3}{|a|}\sup_{(0,T)\times\mathbb{R}}\left\{u^2,\, v^2\right\}. \tag{3.7}$$

We define

$$P_u=\int_0^x udy,\quad P_v=\int_0^x vdy \tag{3.8}$$

Thanks to (3.6), following [12, 17, 27, 40], we can prove that

$$\begin{aligned}&\partial_t(|u-v|)+\partial_x[(au^3-av^3)\mathrm{sign}\,(u-v)]\\ &\qquad -\mathrm{sign}\,(u-v)\,\gamma(P_u-P_v)-\mathrm{sign}\,(u-v)\,b(u-v)-\mathrm{sign}\,(u-v)\,\kappa^2(u^3-v^3)\le 0,\end{aligned}$$

holds in sense of distributions in $(0,\infty)\times\mathbb{R}$, and

$$\begin{aligned}\|u(t,\cdot)-v(t,\cdot)\|_{I(t)} &\le \|u_0-v_0\|_{I(0)}\\ &\quad+\gamma\int_0^t\int_{I(s)}\mathrm{sign}\,(u-v)\,(P_u-P_v)dsdx\\ &\quad+b\int_0^t\int_{I(s)}\mathrm{sign}\,(u-v)\,(u-v)dsdx\\ &\quad+\kappa^2\int_0^t\int_{I(s)}\mathrm{sign}\,(u-v)\,(u^3-v^3)dsdx,\end{aligned} \tag{3.9}$$

for $0 < t < T$, where

$$I(s) = [-R - C(T)(t-s), R + C(T)(t-s)]. \tag{3.10}$$

Observe that

$$\begin{aligned} b\int_0^t\int_{I(s)} \operatorname{sign}(u-v)\,(u-v)dsdx \leq& |b|\int_0^t\int_{I(s)} |u-v|dsdx \\ =& |b|\int_0^t \|u(s,\cdot)-v(s,\cdot)\|_{L^1(I(s))}\,ds. \end{aligned} \tag{3.11}$$

Instead, thanks to (3.6),

$$\begin{aligned} \kappa^2\int_0^t\int_{I(s)} \operatorname{sign}(u-v)\,(u^3-v^3)dsdx \leq& \kappa^2\int_0^t\int_{I(s)} |u^3-v^3|dsdx \\ \leq& C(T)\int_0^t \|u(s,\cdot)-v(s,\cdot)\|_{L^1(I(s))}\,ds. \end{aligned} \tag{3.12}$$

Since

$$|I(s)| = 2R + 2C(T)(t-s) \leq 2R + 2C(T)t \leq C(T), \tag{3.13}$$

due to (3.8),

$$\begin{aligned} \gamma\int_0^t\int_{I(s)} \operatorname{sign}(u-v)\,(P_u-P_v)dsdx \leq& |\gamma|\int_0^t\int_{I(s)} |P_u-P_v|dsdx \\ \leq& |\gamma|\int_0^t\int_{I(s)} \Big(\Big|\int_0^x |u-v|dy\Big|\Big)dsdx \\ \leq& |\gamma|\int_0^t\int_{I(s)} \Big(\Big|\int_{I(s)} |u-v|dy\Big|\Big)dsdx \\ =& |\gamma|\int_0^t |I(s)|\,\|u(s,\cdot)-v(s,\cdot)\|_{L^1(I(s))}\,ds \\ \leq& C(T)\int_0^t \|u(s,\cdot)-v(s,\cdot)\|_{L^1(I(s))}\,ds. \end{aligned} \tag{3.14}$$

Considered the following function,

$$G_1(t) = \|u(t,\cdot)-v(t,\cdot)\|_{I(t)}, \quad t \geq 0. \tag{3.15}$$

It follows from (3.9), (3.11), (3.12) and (3.14) that

$$G_1(t) \leq G_1(0) + C(T)\int_0^t G_1(s)ds. \tag{3.16}$$

The Gronwall inequality and (3.15) give

$$\|u(t,\cdot)-v(t,\cdot)\|_{L^1(-R,R)} \leq e^{C(T)t}\,\|u_0-v_0\|_{L^1(-R-C(T)t,R+C(T)t)},$$

that is (1.14).

□

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