Research article

Existence and nonexistence of global solutions to the Cauchy problem of the nonlinear hyperbolic equation with damping term

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Abstract: This paper concerns with the Cauchy problem for two classes of nonlinear hyperbolic equations with double damping terms. Firstly, by virtue of the Fourier transform method, we prove that the Cauchy problem of a class of high order nonlinear hyperbolic equation admits a global smooth solution \( u(x, t) \in C^\infty(\mathbb{R}^n \times (0, T]; H^\infty) \cap C([0, T]; H^3) \cap C^1([0, T]; H^{-1}) \) as long as initial value \( u_0 \in W^{4,1}(\mathbb{R}) \cap H^3, u_1 \in L^1(\mathbb{R}) \cap H^{-1} \). Moreover, we give the sufficient conditions on the blow-up of the solution of a nonlinear damped hyperbolic equation with the initial value conditions in finite time and an example.

Keywords: nonlinear damped hyperbolic equation; Cauchy problem; Fourier transform; global smooth solution; blow-up

Mathematics Subject Classification: 35K70, 35K61, 35L30, 35L75

1. Introduction

In 1997, Banks et al. [1] established a class of nonlinear damped hyperbolic equation

\[
\begin{align*}
 w_{tt} + A_1 w + A_2 w_t + \sum_{i=1}^{n} g_i(Nw) &= f(t), \\
 w(0) &= \varphi_0, \quad w_t(0) = \varphi_1.
\end{align*}
\]

(1.1)

As a model it describes the motion of a neo-Hooken elastomer rod with internal damping, where \( A_2 w_t \) is the exact form of the internal dynamic damping mechanisms in elastomers, \( A_1, A_2, N \) and \( f \) satisfy certain assumptions.

When \( A_1 = A_2 = -\frac{\partial^2}{\partial x^2}, N = -\frac{\partial}{\partial x} \), Equation (1.1) becomes

\[
u_{tt} - u_{xx} - u_{xxx} = g(u_x)x + f.
\]

(1.3)
When $A_1 = \frac{\partial^4}{\partial x^4}$, $A_2 = -\frac{\partial^2}{\partial x^2}$, $N = -\frac{\partial}{\partial x}$, $f = 0$, Equation (1.1) becomes

$$u_{tt} - u_{xxxx} + u_{xxxxx} = g(u_x)_x.$$  (1.4)

Equation (1.4) describes the propagation of the wave in the medium with the dispersion effect, and it is connected with the equations in [4]–[10]. In [11], Yang et al. proved the well-posedness of Cauchy problem for the nonlinear beam system (1.4). When $g(s) = s^n$, $n \geq 2$, they proved the global existence of smooth solutions as long as initial data $\varphi \in L^2(\mathbb{R}) \cap H^2(\mathbb{R})$, $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. In [12], Chen et al. considered Equation (1.4) with different initial boundary conditions. They proved the existence and uniqueness of the global generalized solutions and the global classical solutions with Galerkin method.

When $A_1 = A_2 = \frac{\partial^4}{\partial x^4}$, $N = -\frac{\partial^2}{\partial x^2}$, Equation (1.1) becomes

$$u_{tt} + u_{xxxx} + u_{xxxxx} = g(u_{xx})_x + f.$$  (1.5)

Equation (1.5) models the vibration of a nonlinear damped beam with fixed boundary, taking account of the internal material damping. Banks et al. [13] established the existence and uniqueness of the global weak solutions to the initial boundary value problem of Equation (1.5). Later, Ackleh et al. [14] studied such system (1.5) to find the existence of weak solutions of the mixed problem in a bounded domain. In [15], Chen et al. gave the sufficient conditions of blow-up result for a nonlinear damped hyperbolic equation. Further generalizations are also given in [16]–[19] and the references therein.

When $A_1 = \frac{\partial^4}{\partial x^4}, A_2 = -\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}, N = -\frac{\partial}{\partial x}, f = 0$, Equation (1.1) becomes

$$u_{tt} - u_{xxxx} + u_{xxxxx} = g(u_{xx})_x, \quad x \in \Omega, \quad t > 0.$$  (1.6)

Recently, Yu et al. [20] established the existence and nonexistence of the global weak solutions to the initial boundary value problem of a nonlinear beam equation with double damping terms (1.6) provided that

$$-C_2|\xi|^2 - C_3 \leq G(\xi) = \int_0^\xi g(\tau)d\tau \leq \frac{1}{2}(1 - \varepsilon)|\xi|^2 + C_1, \quad 0 < \varepsilon < 1$$  (1.7)

$$|g(\xi)| \leq \tilde{C}_1|\xi| + \tilde{C}_2, \quad g'(\xi) \geq 0, \quad \forall \xi \in \mathbb{R}.$$  (1.8)

Obviously, conditions (1.7) and (1.8) imply that the growth order of the nonlinear term $g(s)$ is not more than 1. The reason for the strong assumptions (1.7) and (1.8) lie in that it is very difficult to dominate the effect of the nonlinear term $g(u_{xx})_x$ by standard a priori estimate technology.

When $A_1 = \frac{\partial^4}{\partial x^4}, A_2 = -\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}, N = -\frac{\partial}{\partial x}, f = 0$, Equation (1.1) becomes

$$u_{tt} - k_1u_{xxxx} + u_{xxxxx} = g(u_x)_x, \quad x \in \mathbb{R}, \quad t > 0,$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R},$$  (1.9)  (1.10)
where \( u(x, t) \) denotes the unknown function, \( k_1 \) is a positive constant, \( g(s) \) is a given nonlinear function, \( u_{xxx} \) denotes the strong material damping and \( u_{xt} \) represents the internal dynamic damping.

As far as we know, there is little research on analysis of Equation (1.9) with material damping and internal dynamic damping at the same time. In this case, what happens to the existence and nonexistence of global solution to the problem (1.9)-(1.10) remain open.

When \( A_1 = \frac{\partial^4}{\partial x^4} + \frac{\partial^6}{\partial x^6}, A_2 = -\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}, N = -\frac{\partial}{\partial x}, f = 0 \). If \( g(s) = s^n \), where \( n \geq 2 \) is an integer, Equation (1.1) becomes

\[
 u_{tt} - k_1 u_{xtx} + u_{xxxx} + u_{xxx} + a^2 u_{s,s} = (a^2 u_s)_x, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.11}
\]

where \( a^2 u_{s,s} \) is a “good” regular term. Obviously, \( g(s) \) is monotone if and only if \( n \) is an odd number. For this kind of \( g \), does problem (1.11), (1.10) have any global solutions for initial data belonging to suitably chosen functional spaces? On the other hand, if \( g(s) \) is not monotone and \( g'(s) \) is not bounded below, say \( g(s) = s^{2m} \) (where \( m \geq 1 \) is an integer), does the initial value problem (1.11), (1.10) have any global solutions? The question is interesting and open.

This paper is organized as follows. In Section 2, the main results are stated. In Section 3, we prove the existence of global smooth solutions for the Cauchy problem (1.11), (1.10) by the Fourier transform method. In Section 4, using the modified concavity method, the sufficient conditions of blow-up of the solution for the Cauchy problem (1.9)–(1.10) will be given and we give an example to examine Theorem 2.3.

2. Statement of main results

For problem (1.9)–(1.10), we have the following theorem.

**Theorem 2.1.** Suppose that \( g \in C^2(\mathbb{R}), |g(s)| \leq K_1|s|^q, |g'(s)| \leq K_2|s|^{q-1} \) etc., where \( q \geq 2 \) is natural number and \( K_1, K_2 \) are positive constants. If \( u_0 \in H^0(\mathbb{R}) \) and \( u_1 \in H^1(\mathbb{R}) \), then there is a \( T_1 > 0 \) and the Cauchy problem (1.9)–(1.10) admits a local generalized solution \( u(x, t) \) in \([0, T_1] \times \mathbb{R}\).

This theorem can be proven by the method in [17], we can prove that the periodic boundary value problem admits a local generalized solution by Galerkin method and the compactness theorem. Then using the sequence of the periodic boundary value problem, we can obtain that the Cauchy problem (1.9)-(1.10) has a local generalized solution. Now, we state our main results as follows

**Theorem 2.2.** Suppose that \( u_0 \in W^{4,1}(\mathbb{R}) \cap H^3(\mathbb{R}), u_1 \in L^1(\mathbb{R}) \cap H^{-1}(\mathbb{R}) \). If \( a \neq 0 \), then for any \( T > 0 \), Cauchy problem (1.11), (1.10) admits a global smooth solution \( u(x, t) \in C^\infty([0, T]; H^\infty(\mathbb{R})) \cap C([0, T]; H^3(\mathbb{R})) \cap C^1([0, T]; H^{-1}(\mathbb{R})).

**Remark 2.1.** Theorem 2.2 shows that when excitations occur in the subspace \((W^{4,1}(\mathbb{R}) \cap H^3(\mathbb{R})) \times (L^1(\mathbb{R}) \cap H^{-1}(\mathbb{R})) \subset H^3(\mathbb{R}) \times H^{-1}(\mathbb{R})\), the orbits of the related dynamical system globally exist in the phase space \( H^3(\mathbb{R}) \times H^{-1}(\mathbb{R})\).

**Theorem 2.3.** Suppose that \( u_0 \in H^3(\mathbb{R}), u_1 \in L^2(\mathbb{R}), G(u_{0x}) \in L^1(\mathbb{R}) \) and there exists a constant \( \gamma > 0 \) such that

\[
 s g(s) \leq 2(1 + 2\gamma) G(s), \quad s \in \mathbb{R}, \tag{2.1}
\]

where \( G(s) = \int_0^s g(\tau)d\tau \).

Then the solution \( u(x, t) \) of the Cauchy problem (1.9)–(1.10) blows up in finite time if one of the following conditions holds true:
(i) $E(0) < 0$;
(ii) $E(0) = 0$, $2(u_0, u_1) > \frac{1}{r}(\|u_{0x}\|^2 + k_1\|u_{0x}\|^2)$;
(iii) $E(0) > 0$, $(u_0, u_1) > \left[ E(0)(\|u_0\|^2 + T_0k_1\|u_{0x}\|^2 + T_0\|u_{0x}\|^2) \right]^{\frac{1}{2}}$,
where $E(0) = \|u_1\|^2 + \|u_{0x}\|^2 + 2 \int_\mathbb{R} G(u_{0x})dx$.

3. Existence of the global smooth solutions for the Cauchy problem (1.11), (1.10)

In this section, we establish the existence result for the Cauchy problem (1.11), (1.10) under initial value $u_0 \in W^{4,1}(\mathbb{R}) \cap H^3(\mathbb{R}), u_1 \in L^1(\mathbb{R}) \cap H^{-1}(\mathbb{R})$.

We use the following abbreviations: $\| \cdot \|_{L_p} = \| \cdot \|_{L_p(\mathbb{R})} (1 \leq p < +\infty)$ denotes usual $L^p$ norm, $\| \cdot \| = \| \cdot \|_{L^2(\mathbb{R})}$ denotes the $L^2$-inner product, and equip the Sobolev space $H^l(\mathbb{R})$ with the norm

$$\|f\|_{H^l} = (\int_\mathbb{R} (1 + |\xi|^2)^l |\hat{f}(\xi)|^2 d\xi)^{\frac{1}{2}},$$

for each real number $l$, where $\hat{f}(\xi) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} f(x)e^{-i\xi x}dx$. And we equip the space

$$W^{4,1}(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) | \frac{\partial f}{\partial x^4} \in L^1(\mathbb{R}), i = 1, \ldots, 4 \right\}$$

with the norm

$$\|f\|_{W^{4,1}} = \sum_{i=0}^4 \left\| \frac{\partial f}{\partial x^i} \right\|_{L^1}.$$

Taking the sequences $\{u_0^{(j)}\}_{j=1}^{\infty}, \{u_1^{(j)}\}_{j=1}^{\infty}$ in $C^\infty_0(\mathbb{R})$ such that

$$\|u_0^{(j)} - u_0\|_{W^{4,1}} + \|u_0^{(j)} - u_0\|_{W^{4,1}} \to 0, \quad \|u_1^{(j)} - u_1\|_{L^1} \to 0 \quad as \ j \to \infty. \quad (3.1)$$

Using the properties of Fourier transform derivative, we obtain

$$(1 + |\xi|^4)|\hat{u}_0^{(j)}(\xi) - \hat{u}_0(\xi)| \leq C\|u_0^{(j)} - u_0\|_{W^{4,1}} \to 0 \quad as \ j \to \infty, \quad (3.2)$$

$$|\hat{u}_1^{(j)}(\xi) - \hat{u}_1(\xi)| \leq C\|u_1^{(j)} - u_1\|_{L^1} \to 0 \quad as \ j \to \infty, \quad (3.3)$$

uniformly on $\xi \in \mathbb{R}$, where $C$ denotes positive constant. Hence

$$(1 + |\xi|^4)|\hat{u}_0^{(j)}(\xi)| \leq C_0, \quad |\hat{u}_1^{(j)}(\xi)| \leq C_1, \quad \xi \in \mathbb{R}, \quad (3.4)$$

where and in the sequel $C_i (i = 0, 1, \cdots)$ denotes positive constants independent of $j$ and $\xi$.

Let $u^{(0)}(x, t) \in C^\infty((0, T]; C^\infty(\mathbb{R}))$ be the solution of the following linear problem

$$Lu^{(0)} = u^{(0)}_{tt} - k_1u^{(0)}_{xxx} + u^{(0)}_{xxxx} + u^{(0)}_{xxxx} + \alpha^2u^{(0)}_{x} + b^2u^{(0)} = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (3.5)$$

$$u^{(0)}(x, 0) = u_0^{(0)}(x), \quad u_1^{(0)}(x, 0) = u_1^{(0)}(x), \quad x \in \mathbb{R}, \quad (3.6)$$
where $u_0^0(x), u_1^0(x) \in C(\mathbb{R}), u^{(j)}(x, t) (j = 1, 2, \ldots)$ be the solution of the following nonlinear problem

\[
Lu^{(j)} = [(u_{x}^{(j-1)})^n]_x + b^2 u^{(j-1)}, \quad x \in \mathbb{R}, \ t > 0,
\]

\[
u^{(j)}(x, 0) = u_0^{(j)}(x), \ u_1^{(j)}(x, 0) = u_1^{(j)}(x), \ x \in \mathbb{R},
\]

where $a^2(4 - k_1^2) = 1$ and $k_1 < 2, \ u_{w\xi} = \frac{d^2 u}{d\xi^2}.$

Taking the Fourier transform of (3.7)-(3.8), it follows that

\[
\hat{u}^{(j)}(\xi, t) + (k_1\xi^2 + \xi^4)\hat{u}^{(j)}(\xi, t) + (a^2\xi^8 + \xi^4 + b^2)\hat{u}^{(j)}(\xi, t)
\]

\[= i\xi F[(u_x^{(j-1)})^n](\xi, t) + b^2 \hat{u}^{(j-1)}(\xi, t),
\]

\[
\hat{u}^{(j)}(\xi, 0) = \hat{u}_0^{(j)}(\xi), \ \hat{u}_1^{(j)}(\xi, 0) = \hat{u}_1^{(j)}(\xi), \ x \in \mathbb{R}.
\]

Therefore,

\[
\hat{u}^{(j)}(\xi, t) = e^{-i(k_1\xi^2 + \xi^4)t}\left[(\cos(\omega(\xi)t + \frac{1}{2}(k_1\xi^2 + \xi^4)\omega^{-1}(\xi)\sin(\omega(\xi)t)\hat{u}_0^{(j)}(\xi)
\]

\[+ \omega^{-1}(\xi)\sin(\omega(\xi)t)\cdot \hat{u}_1^{(j)}(\xi)] + \alpha(\xi, t) + \beta(\xi, t),
\]

where $\omega(\xi) = \left[(4a^2 - 1)\xi^8 - 2k_1\xi^6 + (4 - k_1^2)\xi^4 + 4b^2\right]^{1/2},$ and

\[
\alpha(\xi, t) = i\xi\omega^{-1}(\xi) \int_{0}^{t} e^{-\frac{1}{2}(k_1\xi^2 + \xi^4)(t-\tau)} F[(u_x^{(j-1)})^n](\xi, \tau) \sin(\omega(\xi)(t-\tau))d\tau,
\]

\[
\beta(\xi, t) = b^2 \omega^{-1}(\xi) \int_{0}^{t} \hat{u}^{(j-1)}(\xi, \tau)e^{-\frac{1}{2}(k_1\xi^2 + \xi^4)(t-\tau)} \sin(\omega(\xi)(t-\tau))d\tau.
\]

In order to prove Theorem 2.2, we can prove the following lemma.

**Lemma 3.1.** For any $T > 0,$

\[
|\hat{u}^{(j)}(\xi, t)| \leq CH^{(j)}(\xi)e^{-\frac{1}{2}\xi^4t}, \ 0 \leq t \leq T, \ j = 0, 1, \ldots,
\]

where $H^{(j)}(\xi) = |\hat{u}_0^{(j)}(\xi)| + (1 + \xi^2)^{-2}|\hat{u}_1^{(j)}(\xi)| + (1 + |\xi|^4)^{-1},$ $C$ denotes positive constant depending only on $T.$

**Proof.** From (3.4), it follows that

\[
(1 + \xi^4)^{-1} \leq H^{(j)}(\xi) \leq C_2(1 + \xi^4)^{-1}, \ \xi \in \mathbb{R}.
\]

Now, we prove (3.13) by induction method on $j.$ When $j = 0, \alpha(\xi, t) = \beta(\xi, t) = 0,$ and

\[
e^{-\frac{1}{2}(k_1\xi^2 + \xi^4)t} \leq e^{-\frac{1}{2}|k_1|\xi^2t},
\]

\[
\frac{1}{2}(k_1\xi^2 + \xi^4)\omega^{-1}(\xi) \leq C, \ C_3(1 + \xi^4) \leq \omega(\xi) \leq C_4(1 + \xi^4).
\]

From (3.14)–(3.16), it follows that (3.13) is valid.

Assume that (3.13) is valid for $0 < j \leq j_1.$ In order to prove that (3.13) holds true for $j = j_1 + 1,$ we first prove

\[
|F[(u_x^{(j_1-1)})^n](\xi, t)| \leq Ce^{-\frac{1}{2}\xi^4t}, \ t \in [0, T].
\]
In fact, when \( n = 2 \), noting that \(|\xi - \eta| + |\eta| \geq |\xi|\), we have
\[
\|u_x^{(j-1)}(\cdot, t)\| = \|\hat{u}_x^{(j-1)}(\cdot, t)\| \leq C\|\xi H^{(j-1)}(\xi)\| \leq C \left( \int_\mathbb{R} \frac{\xi^2}{(1 + |\xi|^4)^{\frac{1}{2}}} d\xi \right)^{\frac{1}{2}} \leq C,
\]
(3.18)
together with (3.13)(\( j \leq j_1 \)) and Hölder inequality, implies that
\[
|\mathcal{F}[u_x^{(j-1)}]^{(n)}(\xi, t)| = \frac{1}{\sqrt{2\pi}} |(\hat{u}_x^{(j-1)} * \hat{u}_x^{(j-1)})(\xi, t)|
\leq C e^{-\frac{1}{2} |\xi|^4} |(\xi - \eta)H^{(j-1)}(\xi - \eta)||\eta H^{(j-1)}(\eta)|
\leq C e^{-\frac{1}{2} |\xi|^4}, t \in [0, T],
\]
(3.19)
where and in the sequel * denotes convolution. Assume that (3.17) holds true for \( n \leq n_1 \), which implies
\[
\|u_x^{(j-1)}(\cdot, t)\| = \|\mathcal{F}[u_x^{(j-1)}]^{(n)}(\cdot, t)\| < \infty, t \in [0, T].
\]
When \( n = n_1 + 1 \), noting that
\[
\int_\mathbb{R} |\eta H^{(j)}(\eta)| d\eta \leq C \int_\mathbb{R} \frac{|\eta|}{1 + \eta^2} d\eta \leq C,
\]
(3.20)
from (3.13)(\( j \leq j_1 \), (3.17)(\( n \leq n_1 \)) and (3.20), we obtain
\[
|\mathcal{F}[u_x^{(j-1)}]^{(n)}(\xi, t)| = \frac{1}{\sqrt{2\pi}} |(\hat{u}_x^{(j-1)} * \mathcal{F}[u_x^{(j-1)}]^{(n-1)})(\xi, t)|
\leq C e^{-\frac{1}{2} |\xi|^4} \int_\mathbb{R} |\eta H^{(j-1)}(\eta)| d\eta
\leq C e^{-\frac{1}{2} |\xi|^4}, t \in [0, T].
\]
(3.21)
Therefore, (3.17) holds true.

By (3.11), (3.12), (3.13)(\( j \leq j_1 \), (3.14) and (3.17), we see that
\[
|\alpha(\xi, t)| \leq |\xi| \omega^{-1}(\xi) \int_0^t e^{-\frac{1}{2}(k_1 |\xi|^2 + \xi^4)(\tau - t)} \left| \mathcal{F}[(u_x^{(j-1)})^{(n)}]^{(n)}(\xi, \tau) \right| d\tau
\leq C |\xi| \omega^{-1}(\xi) J(\xi, t),
\]
(3.22)
where
\[
|\beta(\xi, t)| \leq b^2 \omega^{-1}(\xi) (1 + \xi^4)^{-1} J(\xi, t),
\]
(3.23)
where
\[
J(\xi, t) = \int_0^t e^{-\frac{1}{2}(\xi^4(\tau - t) - \frac{k_1}{|\xi|^2})} d\tau \leq \begin{cases} \frac{2(e^{-\frac{1}{2} \xi^4} - e^{-\frac{1}{2} \xi^4})}{t}, & \xi \geq \xi_0 \\ \frac{2}{k_1(\frac{t}{\xi_0})^{\frac{1}{4}}}, & \xi < \xi_0\end{cases}
\]
Taking \( \xi_0 \) large enough such that
\[
e^{-\frac{1}{2} |\xi|^4}(1 + \xi^4)|\alpha(\xi, t)| \leq \begin{cases} \frac{2}{k_1(\frac{t}{\xi_0})^{\frac{1}{4}}}, & \xi \geq \xi_0 \\ e^{-\frac{1}{2} \xi_0^4 T} \xi_0 T \leq C, & \xi < \xi_0\end{cases}
\]
Lemma 3.2. For any $\delta: 0 < \delta < T$,
\[
\left| \frac{\partial^m}{\partial t^m} \hat{u}^{(j)}(\xi, t) \right| \leq A(1 + \xi^4)^m H^{(j)}(\xi)e^{-\frac{\xi^4}{T}}, t \in [\delta, T], m = 0, 1, \ldots,
\]  
where and in the sequel $A$ denotes positive constants depending only on $\delta, m$ and $T$.

Proof. when $m = 0$, (3.13) implies (3.27). Assume that (3.27) holds true for $m \leq m_1$. In the following we prove that (3.27) holds true for $m = m_1 + 1$. Now, we need the facts
\[
\left| \frac{\partial^l}{\partial t^l} F[(u_t^{(j-1)})^n](\xi, t) \right| \leq A(1 + \xi^4)^l e^{-\frac{\xi^4}{T}}, t \in [\delta, T], l = 0, 1, \ldots, m - 2.
\]
In fact, when \( n = 2 \), by (3.27)(\( m \leq m_1 \)) we obtain

\[
\left| \frac{\partial^l}{\partial t^l} \mathcal{F}[(u_x^{(j-1)})^2](\xi, t) \right| \leq \sum_{s=0}^{l} C_s^l \int_{\mathbb{R}} |\xi - \eta| \frac{\partial^{l-s}}{\partial t^{l-s}} \hat{u}^{(j-1)}(\xi - \eta, t) \cdot |\eta| \frac{\partial^{l-s}}{\partial t^{l-s}} \hat{u}^{(j-1)}(\eta, t) d\eta
\]

\[
\leq \sum_{s=0}^{l} C_s^l A \int_{\mathbb{R}} |\xi - \eta|(1 + (\xi - \eta)^4) e^{-\frac{1}{t-\eta} H^{(j-1)}(\xi - \eta)} \cdot |\eta|(1 + \eta^4)^{l-s} e^{-\frac{1}{t-\eta} H^{(j-1)}(\eta)} d\eta.
\]

(3.29)

Noting that

(i) When \( |\eta| \leq 2|\xi| \), we have \( |\xi - \eta| \leq 3|\xi| \), and

\[
(1 + (\xi - \eta)^4)^{(1 + \eta^4)^{l-s}} \leq C(1 + \xi^4)^l.
\]

(ii) When \( |\eta| > 2|\xi| \), we have \( |\xi - \eta| > |\xi| \), and

\[
e^{-\frac{1}{t-\eta} H^{(j-1)}(\xi - \eta)} \cdot |\eta|(1 + \eta^4)^{l-s} e^{-\frac{1}{t-\eta} H^{(j-1)}(\eta)} d\eta
\]

\[
\leq C e^{-\frac{1}{t-\eta} H^{(j-1)}(\xi - \eta)} \cdot |\eta|(1 + \eta^4)^{l-s} e^{-\frac{1}{t-\eta} H^{(j-1)}(\eta)} d\eta
\]

\[
\leq C t^{-l}(1 + \xi^4)^l
\]

\[
\leq A(1 + \xi^4)^l, \quad t \in [\delta, T].
\]

Substituting above inequalities into (3.29) and using Hölder inequality, we obtain

\[
\left| \frac{\partial^l}{\partial t^l} \mathcal{F}[(u_x^{(j-1)})^2](\xi, t) \right|
\]

\[
\leq \sum_{s=0}^{l} C_s^l A \left( \int_{-2|\xi|}^{2|\xi|} + \int_{-2|\xi|}^{2|\xi|} \right) |\xi - \eta|(1 + (\xi - \eta)^4)^{l-s} e^{-\frac{1}{t-\eta} H^{(j-1)}(\xi - \eta)} H^{(j-1)}(\xi - \eta) \cdot |\eta|(1 + \eta^4)^{l-s} e^{-\frac{1}{t-\eta} H^{(j-1)}(\eta)} d\eta
\]

\[
\leq \sum_{s=0}^{l} C_s^l A(1 + \xi^4)^l e^{-\frac{1}{t-\eta} H^{(j-1)}(\xi - \eta)} \cdot |\eta| H^{(j-1)}(\xi - \eta) d\eta
\]

\[
\leq \sum_{s=0}^{l} C_s^l A(1 + \xi^4)^l e^{-\frac{1}{t-\eta} H^{(j-1)}(\xi - \eta)}(1 + \eta^4)^{l-s} e^{-\frac{1}{t-\eta} H^{(j-1)}(\eta)} d\eta
\]

\[
\leq A(1 + \xi^4)^l e^{-\frac{1}{t-\eta} H^{(j-1)}(\xi - \eta)}, \quad t \in [\delta, T], l = 0, 1, \ldots, m - 2.
\]

Assume that (3.28) holds true for \( n \leq n_1 \), when \( n = n_1 + 1 \), from (3.28)(\( n \leq n_1 \)), (3.27)(\( m \leq m_1 \)), it
follows that
\[
\left| \frac{\partial^l}{\partial t^l} \mathcal{F}[(u_s^{(j-1)})^\eta](\xi, t) \right| \\
= \left| \frac{\partial^l}{\partial t^l} \mathcal{F}[(u_s^{(j-1)}) \cdot (u_s^{(j-1)})^\eta-1](\xi, t) \right| \\
= \frac{1}{\sqrt{2\pi}} \frac{\partial^l}{\partial t^l} \mathcal{F}[(u_s^{(j-1)})^\eta-1](\xi, t) \\
\leq \left| \frac{\partial^l}{\partial t^l} \int (\xi - \eta) \hat{u}^{(j-1)}(\xi - \eta, t) \cdot \mathcal{F}[(u_s^{(j-1)})^\eta-1](\eta, t) d\eta \right| \\
\leq \sum_{i=0}^l C_i \int |(\xi - \eta) \frac{\partial^l}{\partial t^l} \hat{u}^{(j-1)}(\xi - \eta, t) \cdot \frac{\partial^{l-s}}{\partial t^{l-s}} \mathcal{F}[(u_s^{(j-1)})^\eta-1](\eta, t) d\eta \\
\leq \sum_{i=0}^l C_i A \int |(\xi - \eta)(1 + (\xi - \eta)^4) e^{-\frac{\xi}{2} H^{(j-1)}(\xi - \eta)} \\
\cdot (1 + \eta^4)^{l-s} e^{-\frac{\xi}{2} H^{(j-1)} d\eta} \\
\leq A(1 + \xi^4)^{l-s} e^{-\frac{\xi}{2} H^{(j-1)}} d\eta, \quad t \in [\delta, T], \quad l = 0, 1, \ldots, m - 2.
\]

Therefore, (3.28) holds true.

From (3.27) \((m \leq m_1)\) and (3.28), it follows that
\[
|h_i(\xi, t)| + |g_i(\xi, t)| \\
\leq C(1 + \xi^4) e^{-\frac{1}{4} (\xi^2 + \xi^4)^4} \\
\leq C(1 + \xi^4)^4 e^{-\frac{\xi}{4} H^{(j-1)}} d\eta, \quad t \in [0, T], \quad i = 0, 1, \ldots, m - 2.
\]

Similarly,
\[
\left| \frac{\partial^{m-2}}{\partial t^{m-2}} R_m(\xi, t) \right| \\
\leq C(1 + \xi^4) \omega^{-1}(\xi) \left| \xi \frac{\partial^{m-2}}{\partial t^{m-2}} \mathcal{F}[(u_s^{(j-1)})^\eta](\xi, t) \right| + \left| \frac{\partial^{m-2}}{\partial t^{m-2}} \hat{u}^{(j-1)}(\xi, t) \right| \\
\leq C e^{-\frac{\xi}{2} |\xi|}(1 + \xi^4)^m H^{(j-1)}(\xi), \quad t \in [\delta, T].
\]

\[
\left| \frac{\partial^s}{\partial t^s} R_{m-s}(\xi, t) \right| \\
\leq C e^{-\frac{\xi}{2} |\xi|}(1 + \xi^4)^m H^{(j)}(\xi), \quad t \in [\delta, T], \quad s = 2, \ldots, m - 3
\]

\[
|R_m(\xi, t)| \leq C(1 + \xi^4)^m \omega^{-1}(\xi) |\xi| \mathcal{F}[(u_s^{(j-1)})^\eta](\xi, t) + \hat{u}^{(j-1)}(\xi, t) \\
+ \sum_{l=1}^{m-2} \left| \frac{\partial^l}{\partial t^l} R_{m-l}(\xi, t) \right| \\
\leq C e^{-\frac{\xi}{2} |\xi|}(1 + \xi^4)^m \omega^{-1}(\xi) |\xi| + H^{(j-1)}(\xi) \\
+ C e^{-\frac{\xi}{2} |\xi|}(1 + \xi^4)^m H^{(j)}(\xi) \\
\leq A e^{-\frac{\xi}{2} |\xi|}(1 + \xi^4)^m H^{(j)}(\xi), \quad t \in [\delta, T].
\]
Therefore, (3.27) holds true.

**Proof of Theorem 2.2.**

For any $\delta > 0$, we can extract a subsequence from $\{u^{(j)}\}$, still denoted by $\{u^{(j)}\}$, such that

$$u^{(j)} \to u \text{ in } C^m([\delta, T]; H^l(\mathbb{R})) \text{ as } j \to \infty,$$  

(3.36)

where $j$ and $l$ are positive integers. Moreover, $u$ satisfies Equation (1.11) for $t > 0$.

In fact, from Lemma 3.1 and Lemma 3.2, it deduces that

$$u^{(j)}(\cdot, t) \to u(\cdot, t) \text{ in } H^l(\mathbb{R}),$$

(3.37)

where $\delta$, $\lambda$, $\mu$, and $\nu$ are positive constants depending on $\delta$, $\lambda$, and $\nu$. Using Arzelà–Ascoli theorem, from (3.37)–(3.38) we arrive at (3.36) holds true.

Letting $j \to \infty$ in Equation (3.7), by the arbitrariness of $\delta$, we obtain that $u$ satisfies Equation (1.11) for $t > 0$.

Now we will prove

$$u^{(j)} \to u \text{ in } C([0, T]; H^3(\mathbb{R})) \cap C^1([0, T]; H^{-1}(\mathbb{R})) \text{ as } j \to \infty,$$  

(3.39)

namely,

$$u(x, 0) = u_0(x) \text{ in } H^3(\mathbb{R}), \quad u_t(x, 0) = u_t^1(x) \text{ in } H^{-1}(\mathbb{R}).$$

In fact, we extract two subsequences from $\{u_0^{(j)}\}$ and $\{u_t^{(j)}\}$, respectively, we can still denote $\{u_0^{(j)}\}$ and $\{u_t^{(j)}\}$, from (3.1), it follows that

$$||u_0^{(j)} - u_0^{(j-1)}||_{W^{1,1}} + ||u_t^{(j)} - u_t^{(j-1)}||_{H^l} \leq 2^{-j}, \quad ||u_t^{(j)} - u_t^{(j-1)}||_{L^1} \leq 2^{-j}.$$  

(3.40)

Furthermore, we obtain

$$\sup_{0 \leq t \leq T} ||\tilde{u}_0^{(j)}(\xi, t) - \tilde{u}_0^{(j-1)}(\xi, t)|| \leq C[2^{1-j}(H^{(j)}(\xi) + H^{(j-1)}(\xi)) + G^{(j)}(\xi)],$$  

(3.41)

where

$$G^{(j)}(\xi) = ||\tilde{u}_0^{(j)}(\xi) - \tilde{u}_0^{(j-1)}(\xi)|| + (1 + \xi^2)^{-1}||\tilde{u}_1^{(j)}(\xi) - \tilde{u}_1^{(j-1)}(\xi)||$$

$$\leq \frac{2^{-j}}{1 + \xi^2} + \frac{2^{-j}2^{1-j}}{(1 + \xi^2)^2}$$

$$\leq \frac{2^{1-j}}{1 + \xi^2} \leq 2^{1-j}H^{(j)}(\xi).$$  

(3.42)
Now, we will prove (3.41) holds true by induction method. Indeed, when \( j = 1 \), by (3.13) we see that

\[
\sup_{0 \leq t \leq T} \left| \hat{u}^{(1)}(\xi, t) - \hat{u}^{(0)}(\xi, t) \right| \leq C(H^{(1)}(\xi) + H^{(0)}(\xi)),
\]

(3.43)

that is, (3.41) holds true.

Assume that (3.41) holds true for \( j \leq j_1 \). When \( j = j_1 + 1 \), we deduce from (3.10) that

\[
\left| \hat{u}^{(j)}(\xi, t) - \hat{u}^{(j-1)}(\xi, t) \right|
\leq Ce^{-\frac{1}{2}(k_i \xi^2 + \xi^4) \xi^{-1}}\int_0^t e^{-\frac{1}{2}(k_i \xi^2 + \xi^4) \xi^{-1} \tau} P_j(\xi, \tau) d\tau
\]

\[
+ b^2 \omega^{-1} \int_0^t e^{-\frac{1}{2}(k_i \xi^2 + \xi^4) \xi^{-1} \tau} | \hat{u}^{(j-1)}(\xi, t) - \hat{u}^{(j-2)}(\xi, \tau) | d\tau,
\]

(3.44)

where

\[
P_j(\xi, t) = \left| \mathcal{F}[(u^{(j-1)}_x)^n] - \mathcal{F}[(u^{(j-2)}_x)^n] \right|.
\]

(3.45)

We claim that

\[
\sup_{0 \leq t \leq T} P_j(\xi, t) \leq 2^{1-j} C.
\]

(3.46)

In fact, when \( n = 2 \), we have

\[
\left( \int_{\mathbb{R}} \eta^2 |G^{(j-1)}(\eta)|^2 d\eta \right)^{\frac{1}{2}} \leq 2^{1-j} \left( \int_{\mathbb{R}} \frac{\eta^2}{(1 + \eta^2)^2} d\eta \right)^{\frac{1}{2}} \leq 2^{1-j} C.
\]

(3.47)

From (3.41)\((j \leq j_1)\), (3.13) and H"older inequality, it deduce that

\[
\sup_{0 \leq t \leq T} P_j(\xi, t)
= \sup_{0 \leq t \leq T} \left| (\xi \hat{u}^{(j-1)} - \xi \hat{u}^{(j-2)} + \xi \hat{u}^{(j-2)})(\xi, t) \right|
= \sup_{0 \leq t \leq T} \left| \int_{\mathbb{R}} \eta(\hat{u}^{(j-1)}(\eta, t) - \hat{u}^{(j-2)}(\eta, t))(\xi - \eta) d\eta \right| \cdot | \xi - \eta | (H^{(j-1)}(\eta) + H^{(j-2)}(\eta)) d\eta
\]

\[
\leq \int_{\mathbb{R}} \eta | \xi |^2 (H^{(j-1)} + H^{(j-2)})(\eta) d\eta \leq 2^{1-j} C.
\]

(3.48)

Assume that (3.46) holds true for \( n \leq n_1 \). When \( n = n_1 + 1 \), we deduce from (3.41)(\( j \leq j_1 \)), (3.13),
(3.17) and (3.46) \((n \leq n_1)\) that

\[
\sup_{0 \leq t \leq T} P_j(\xi, t) = \sup_{0 \leq t \leq T} \left| (\xi \hat{u}^{(j-1)} * F[(u_s^{(j-1)})^{n-1}] - \xi \hat{u}^{(j-2)} * F[(u_s^{(j-2)})^{n-1}]) (\xi, t) \right|
\]

\[
\leq \sup_{0 \leq t \leq T} \left| \xi (\hat{u}^{(j-1)} - \hat{u}^{(j-2)}) * F[(u_s^{(j-1)})^{n-1}] + \xi \hat{u}^{(j-2)} * (F[(u_s^{(j-1)})^{n-1}] - F[(u_s^{(j-2)})^{n-1}]) (\xi, t) \right|
\]

\[
\leq C \int_\mathbb{R} |\eta| \left[ 2^{|-j|} (H^{(j-1)} + H^{(j-2)})(\eta) \right] + G^{(j-1)}(\eta) \, d\eta + 2^{1-j} C \int_\mathbb{R} |\xi - \eta| H^{(j-2)}(\xi - \eta) \, d\eta
\]

\[
\leq 2^{1-j} C,
\]

that is, (3.46) holds true.

It follows from (3.41) \((j \leq j_1)\), (3.44)-(3.46) that

\[
\left| \hat{u}(\xi, t) - \hat{u}(\xi, t) \right|
\]

\[
\leq G^{(j)}(\xi) + C \omega^{-1}(\xi) \left[ 2^{1-j}|\xi| + b^2 2^{-j} (H^{(j-1)})(\xi) + f^{(j-2)}(\xi) + G^{(j-1)}(\xi) \right] \cdot Q(\xi, t),
\]

where

\[
Q(\xi, t) = \int_0^t e^{-\frac{1}{2}(t-t)} \, d\tau \leq \left\{ \begin{array}{ll}
\frac{2(1-e^{-\frac{1}{2}t})}{t}, & |\xi| \geq \xi_0 \\
|\xi| < \xi_0
\end{array} \right.
\]

By (3.42)–(3.50), we have

\[
|\xi| \omega^{-1}(\xi) Q(\xi, t) \leq C(1 + \xi^4)^{-1} \leq CH^{(j)}(\xi),
\]

\[
b^2 \omega^{-1}(\xi) H^{(j-1)}(\xi) Q(\xi, t) \leq C(1 + \xi^4)^{-1} \leq CH^{(j)}(\xi),
\]

\[
\omega^{-1}(\xi) G^{(j-1)}(\xi) Q(\xi, t) \leq 2^{1-j} H^{(j)}(\xi), \, t \in [0, T].
\]

Substituting (3.51)–(3.53) into (3.50) yields (3.41).

It follows from (3.41)–(3.42) and (3.14) that

\[
\sup_{0 \leq t \leq T} \left\| u^{(j)}(\cdot, t) - u^{(j-1)}(\cdot, t) \right\|_{H^3}
\]

\[
= \sup_{0 \leq t \leq T} \left\| (1 + \xi^2)^{ \frac{1}{2} } (\hat{u}(\xi, t) - \hat{u}(\xi, t)) \right\|_{L^2}
\]

\[
\leq 2^{1-j} C \left\| (1 + \xi^2)^{ \frac{1}{2} } (H^{(j)}(\xi) + H^{(j-1)}(\xi)) \right\| + \left\| (1 + \xi^2)^{ \frac{1}{2} } G^{(j)}(\xi) \right\|
\]

\[
\leq 2^{1-j} C,
\]

Hence, we can write \( \{u^{(j)}\} \) is a Cauchy sequence in \( C([0, T]; H^3(\mathbb{R})) \) and this implies that

\[
u^{(j)} \to u \text{ in } C([0, T]; H^3(\mathbb{R})) \text{ as } j \to \infty.
\]
Similarly, by (3.51)–(3.53), we obtain
\[
\left| \tilde{u}_t^{(j)}(\xi, t) - \tilde{u}_t^{(j-1)}(\xi, t) \right|
\leq C(1 + \xi^2) \left[ G^{(j)}(\xi) e^{-\frac{1}{2} \langle k_1 \xi^2 + \xi^4 \rangle \gamma t} + \omega^{-1}(\xi) \int_0^t e^{-\frac{1}{2} \langle k_1 \xi^2 + \xi^4 \rangle \gamma (t-\tau)} P_j(\xi, \tau) d\tau \right]
\]
\[
+ b^2 \omega^{-1}(\xi) \int_0^t e^{-\frac{1}{2} \langle k_1 \xi^2 + \xi^4 \rangle \gamma (t-\tau)} |\tilde{u}_t^{(j-1)}(\xi, \tau) - \tilde{u}_t^{(j-2)}(\xi, \tau)| d\tau
\]
\[
\leq C(1 + \xi^4) \left[ G^{(j)}(\xi) + \omega^{-1}(\xi) (2^{1-j} \omega \xi) \right]
\]
\[
+ b^2 (2^{2-j} (H^{(j-1)}(\xi) + H^{(j-2)}(\xi)) + G^{(j-1)}(\xi)) Q(\xi, t)
\]
\[
\leq C(1 + \xi^4) \left[ G^{(j)}(\xi) + 2^{1-j} (H^{(j)}(\xi) + H^{(j-1)}(\xi)) \right], \quad t \in [0, T].
\]

Furthermore
\[
\sup_{0 \leq t \leq T} \left\| u_t^{(j)}(\cdot, t) - u_t^{(j-1)}(\cdot, t) \right\|_{H^{-1}}
\]
\[
= \sup_{0 \leq t \leq T} \left\| (1 + \xi^2)^{-1/2} (\tilde{u}_t^{(j)} - \tilde{u}_t^{(j-1)})(\cdot, t) \right\|
\]
\[
\leq C \left\| (1 + \xi^2)^{3/2} G^{(j)}(\xi) \right\| + 2^{1-j} C \left\| (1 + \xi^2)^{3/2} (H^{(j)}(\xi) + H^{(j-1)}(\xi)) \right\|
\]
\[
\leq C \left\| (1 + \xi^2)^{3/2} (1 + \xi^4)^{-1} \right\| + 2^{1-j} C \left\| (1 + \xi^2)^{3/2} (1 + \xi^4)^{-1} \right\|
\]
\[
\leq 2^{1-j} C.
\]

Hence, we can write \( u_t^{(j)} \) is a Cauchy sequence in \( C([0, T]; H^{-1}(\mathbb{R})) \) and this implies that
\[
u_t^{(j)} \to u_t \text{ in } C([0, T]; H^{-1}(\mathbb{R})) \text{ as } j \to \infty.
\]

Thus, \( u \) is a solution of problem (1.11), (1.10). This completes the proof of Theorem 2.2.

4. Nonexistence of the solution for the Cauchy problem (1.9)–(1.10)

To obtain the blow-up result of the solution to the Cauchy problem (1.9)–(1.10), we introduce the following lemma.

**Lemma 4.1.** ([8]) Suppose that a positive, twice-differentiable function \( \theta(t) \) satisfies the inequality
\[
\theta''(t) \theta(t) - (1 + \gamma) \theta^2(t) \geq -2C_1 \theta'(t) \theta(t) - C_2 \theta^2(t), \quad t > 0,
\]
where \( \gamma > 0 \) and \( C_1, C_2 \geq 0 \) are constants.

(1) If \( C_1 = C_2 = 0, \theta(0) > 0 \) and \( \theta'(0) > 0 \), then there exists \( t_1 \in (0, \frac{m_0}{\gamma \theta'(0)}) \) such that \( \theta(t) \) tends to infinity as \( t \to t_1 \).

(2) If \( C_1 + C_2 > 0, \theta(0) > 0 \) and \( \theta'(0) > -\gamma_2 \gamma^{-1} \theta(0) \), then there exists \( t_1 > 0 \) such that \( \theta(t) \) tends to infinity as \( t \to t_1 \), where \( t_1 \) is bounded above by
\[
\frac{1}{2 \sqrt{C_1^2 + \gamma C_2}} \ln \left( \frac{\gamma_1 \theta(0) + \gamma \theta'(0)}{\gamma_2 \theta(0) + \gamma \theta'(0)} \right)
\]
with \( \gamma_1 = -C_1 + \sqrt{C_1^2 + \gamma C_2} \) and \( \gamma_2 = -C_1 - \sqrt{C_1^2 + \gamma C_2} \).

**Proof of Theorem 2.3.**
Suppose that the maximal time of existence of the solution of the Cauchy problem (1.9)–(1.10) is infinite. The energy functional for the problem (1.9) can be defined as

\[
E(t) = ||u(t)||^2 + ||u_{xx}(t)||^2 + 2k_1 \int_{0}^{t} ||u_{xx}(\tau)||^2 d\tau \\
+ 2 \int_{0}^{t} ||u_{xxx}(\tau)||^2 d\tau + 2 \int_{\mathbb{R}} G(u) dx.
\]

From (4.1) and (1.9), it follows that

\[
\frac{d}{dt} E(t) = \frac{d}{dt} \left[ ||u(t)||^2 + ||u_{xx}(t)||^2 + 2k_1 \int_{0}^{t} ||u_{xx}(\tau)||^2 d\tau \\
+ 2 \int_{0}^{t} ||u_{xxx}(\tau)||^2 d\tau + 2 \int_{\mathbb{R}} G(u) dx \right] \\
= 2 \int_{\mathbb{R}} \left( u_t u_t + u_{xxxx} u_t - k_1 u_{xxx} u_t + u_{xxxxx} u_t - g(u_t) u_t \right) dx \\
= 0.
\]

Integration of (4.2) from 0 to \( t \) leads to

\[
E(t) = E(0).
\]

We now define

\[
\phi(t) = ||u(t)||^2 + k_1 \int_{0}^{t} ||u_{xx}(\tau)||^2 d\tau + \int_{0}^{t} ||u_{xxx}(\tau)||^2 d\tau \\
+ k_1 (T_0 - t)||u_{0xx}||^2 + (T_0 - t)||u_{0xxx}||^2 + \beta(t + t_0)^2,
\]

where \( \beta \geq 0, T_0 \) and \( t_0 \) are positive real numbers to be given later. Hence,

\[
\phi'(t) = 2 \int_{\mathbb{R}} u(t) u_t(t) dx + k_1 ||u_x(t)||^2 + ||u_{xx}(t)||^2 - k_1 ||u_{0xx}(t)||^2 \\
- ||u_{0xxx}(t)||^2 + 2\beta(t + t_0) \\
= 2 \left[ \int_{\mathbb{R}} u(t) u_t(t) dx + k_1 \int_{\mathbb{R}} u_x(\tau) u_{xx}(\tau) d\tau + \beta(t + t_0) \right] \\
+ \int_{0}^{t} \int_{\mathbb{R}} u_{xx}(\tau) u_{xxx}(\tau) d\tau d\tau + \beta(t + t_0) \right].
\]

Then we have

\[
\phi''(t) \\
= 2 \left[ ||u_t(t)||^2 + \int_{\mathbb{R}} u(t) u_{tt}(t) dx + k_1 \int_{\mathbb{R}} u_x(t) u_{xxt}(t) dx + \int_{\mathbb{R}} u_{xx}(t) u_{xxt}(t) dx + \beta \right] \\
= 2 \left[ ||u_t(t)||^2 - ||u_{xx}(t)||^2 - (g(u_x), u_x) + \beta \right].
\]
From (4.4), we can write

\[ \phi'(t)^2 = 4 \left[ \int_R u(t)u_x(t)dx + k_1 \int_0^t \int_R \|u_x(t)\| \|u_x(t)\| d\tau + \int_0^t \int_R \|u_{xx}(t)\| \|u_{xx}(t)\| d\tau \right. \]
\[ \left. + \beta(t + t_0) \right]^2 \]
\[ \leq 4 \left[ \|u(t)\| \|u(t)\| + \left( k_1 \int_0^t \|u_x(t)\|^2 d\tau \right)^{\frac{1}{2}} \left( k_1 \int_0^t \|u_{xx}(t)\|^2 d\tau \right)^{\frac{1}{2}} \right] \]
\[ + \left( \int_0^t \|u_x(t)\|^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^t \|u_{xx}(t)\|^2 d\tau \right)^{\frac{1}{2}} + \sqrt{\beta} \sqrt{\beta(t + t_0)} \right]^2 \]
\[ \leq 4\beta(t) \|u(t)\|^2 + k_1 \int_0^t \|u_x(t)\|^2 d\tau + \int_0^t \|u_{xx}(t)\|^2 d\tau + \beta. \]}

Now, from (4.4)–(4.6), we can obtain that

\[ \phi''(t)\phi(t) - (1 + \gamma)\phi'(t)^2 \]
\[ = 2\phi(t) \left[ \|u_x(t)\|^2 - \|u_{xx}(t)\|^2 - \int_R g(u_x(t))u_x(t)dx + \beta \right] - (1 + \gamma)\phi'(t)^2 \]
\[ \geq 2\phi(t) \left[ \|u_x(t)\|^2 - \|u_{xx}(t)\|^2 - \int_R g(u_x(t))u_x(t)dx + \beta \right] \]
\[ - 2(1 + \gamma) \left( \|u_x(t)\|^2 + k_1 \int_0^t \|u_x(t)\|^2 d\tau + k_1 \int_0^t \|u_{xx}(t)\|^2 d\tau + \beta \right) \]
\[ \geq 2\phi(t)\psi(t), \]

where

\[ \psi(t) = -\|u_{xx}(t)\|^2 - \int_R g(u_x(t))u_x(t)dx - (2\gamma + 1)(\|u_x(t)\|^2 + \beta) \]
\[ - 2(1 + \gamma) \int_0^t (k_1\|u_x(t)\|^2 + \|u_{xx}(t)\|^2) d\tau. \]}

From (4.8) and (1.9), it deduces that

\[ \psi'(t) = -2 \int_R u_x(t)u_{xx}(t)dx - \frac{d}{dt} \int_R g(u_x(t))u_x(t)dx \]
\[ - 2(1 + 2\gamma) \int_R u_x(t)u_{xx}(t)dx - 2(1 + \gamma)(k_1\|u_x(t)\|^2 + \|u_{xx}(t)\|^2) \]
\[ = -\frac{d}{dt} \int_R g(u_x(t))u_x(t)dx + 2(1 + 2\gamma) \int_R g(u_x(t))u_x(t)dx \]
\[ + 4\gamma \int_R u_{xx}(t)u_{xx}(t)dx + 2\gamma\|u_{xx}(t)\|^2 + 2\gamma k_1\|u_x(t)\|^2. \]
Integrating (4.9) from 0 to \( t \) and making use of the assumption (2.1),

\[
\psi(t) \geq \psi(0) + \int_0^t g(u_0)u_0 dx - \int_0^t g(u_x(t))u_x(t)dx \\
+ 2(1 + 2\gamma) \int_0^t G(u_x(t))dx - 2(1 + 2\gamma) \int_0^t G(u_0)dx \\
+ 2\gamma ||u_{xx}(t)||^2 - 2\gamma ||u_{0,xx}||^2 \\
\geq \psi(0) + \int_0^t g(u_0)u_0 dx - 2(1 + 2\gamma) \int_0^t G(u_0)dx - 2\gamma ||u_{0,xx}||^2 \\
\geq -(1 + 2\gamma)(||u_{0,xx}||^2 + ||u_1||^2 + 2 \int_0^T G(u_0)dx + \beta) \\
\geq -(1 + 2\gamma)(E(0) + \beta).
\]

Combining this inequality with (4.7), we obtain

\[
\phi''(t)\phi(t) - (1 + \gamma)\phi'(t)^2 \geq -2(1 + 2\gamma)(E(0) + \beta)\phi(t).
\]

We consider three different cases on the sign of the initial energy \( E(0) \):

(1) If \( E(0) < 0 \), we choose \( \beta = -E(0) > 0 \). Then it follows from (4.11) that

\[
\phi''(t)\phi(t) - (1 + \gamma)\phi'(t)^2 \geq 0.
\]

We take \( t_0 \) in a such way that

\[
2(u_0, u_1) + 2\beta t_0 > \frac{1}{\gamma}(||u_{0,xx}||^2 + k_1||u_{0,xx}||^2),
\]

then \( \phi(0) > 0, \phi'(0) > 0 \). We choose \( T_0 \) such that

\[
\frac{||u_0||^2 + \beta t_0^2}{2\gamma \left( \int_0^T u_0(x)u_1(x)dx + \beta t_0 \right) - ||u_{0,xx}||^2 - k_1||u_{0,xx}||^2} \leq T_0.
\]

According to Lemma 4.1 (1), there exists \( t_1 \), where \( 0 < t_1 \leq \frac{\phi(0)}{\gamma \phi'(0)} = \frac{||u_0||^2 + k_1T_0||u_0||^2 + T_0||u_{0,xx}||^2 + \beta t_0^2}{\gamma \int_0^T u_0(x)u_1(x)dx + \beta t_0} \) such that

\[
\phi(t) \to \infty \text{ as } t \to t_1^-
\]

i.e.

\[
||u(t)||^2 + k_1 \int_0^t ||u_x(\tau)||^2 d\tau + \int_0^t ||u_{xx}(\tau)||^2 d\tau \to \infty \text{ as } t \to t_1^-.
\]

This is a contradiction with the fact that the maximal time of the existence of the solution is infinite.

(2) If \( E(0) = 0 \), we choose \( \beta = 0 \) Then (4.11) becomes

\[
\phi''(t)\phi(t) - (1 + \gamma)\phi'(t)^2 \geq 0.
\]
We take $T_0$ such that
\[ \frac{\|u_0\|^2}{2\gamma \left[ \int_{\mathbb{R}} u_0(x)u_1(x)dx \right] - \|u_{0xx}\|^2 - k_1 \|u_{0x}\|^2} \leq T_0. \]

By considering the assumption (ii), we obtain $\phi(0) > 0, \phi'(0) > 0$. Then according to Lemma 4.1 (1), there exists $t_2$, where $0 < t_2 \leq \frac{\phi(0)}{\gamma \phi'(0)} = \frac{\|u_0\|^2 + k_1 \int_{\mathbb{R}} u_0(x)u_1(x)dx}{\gamma \|u_{0x}\|^2}$, such that
\[ \phi(t) \to \infty \text{ as } t \to t_2^- \]
i.e.
\[ \|u(t)\|^2 + k_1 \int_0^t \|u_x(\tau)\|^2 d\tau + \int_0^t \|u_{xx}(\tau)\|^2 d\tau \to \infty \text{ as } t \to t_2^- . \]

This is a contradiction with the fact that the maximal time of the existence of the solution is infinite.

(3) If $E(0) > 0$, we choose $\beta = 0$, then (4.11) becomes
\[ \phi''(t)\phi(t) - (1 + \gamma)\phi'(t)^2 \geq -2(1 + 2\gamma)E(0)\phi(t). \quad (4.14) \]

We now define the auxiliary function $J(t)$ as follows:
\[ J(t) = -\phi^{-\gamma}(t). \quad (4.15) \]

Now, we compute
\[ J'(t) = \gamma \phi^{-\gamma - 1}(t)\phi'(t). \quad (4.16) \]
and
\[ J''(t) = -\gamma(\gamma + 1)\phi^{-\gamma - 2}(t)\phi'(t)^2 + \gamma \phi^{-\gamma - 1}(t)\phi''(t) \]
\[ = \gamma \phi^{-\gamma - 2}(t) \left[ \phi''(t)\phi(t) - (1 + \gamma)\phi'(t)^2 \right] \quad (4.17) \]
\[ \geq -2\gamma(1 + 2\gamma)E(0)\phi^{-\gamma - 1}(t) \]

By considering the assumption (iii), we deduce
\[ J'(0) = \gamma \phi^{-\gamma - 1}(0)\phi'(0) > 0. \]

It follows the continuity of $J'(t)$ that
\[ J'(t) > 0 \quad (4.18) \]
for some interval near $t = 0$. Let $t^* > 0$ be a maximal time (possibly $t^* = T$) when (4.18) holds on $[0, t^*)$. Multiplying (4.17) by $2J'(t)$, we obtain
\[ \frac{d}{dt} \left[ J'(t) \right]^2 \geq -4\gamma(2\gamma + 1)E(0)\phi^{-\gamma - 1}(t)J'(t) \]
\[ = -4\gamma^2(2\gamma + 1)E(0)\phi^{-2\gamma - 2}(t)\phi'(t) \quad (4.19) \]
\[ = 4\gamma^2 E(0) \frac{d}{dt} \left[ \phi^{-2\gamma - 2}(t) \right], \quad \forall t \in [0, t^*) \]
Integrating (4.19) over \([0, t]\), we obtain
\[
J'(t)^2 \geq J'(0)^2 + 4\gamma^2 E(0)\phi^{-2\gamma-1}(t) - 4\gamma^2 E(0)\phi^{-2\gamma-1}(0)
\geq J'(0)^2 - 4\gamma^2 E(0)\phi^{-2\gamma-1}(0).
\] (4.20)

From the assumption (iii), we obtain
\[
J'(0)^2 - 4\gamma^2 E(0)\phi^{-2\gamma-1}(0) > 0.
\]

Hence, making use of the continuity of \(J'(t)\), we obtain
\[
J'(t) \geq \left[ J'(0)^2 - 4\gamma^2 E(0)\phi^{-2\gamma-1}(0) \right]^{1/2}, \quad \forall t \in [0, t^*).
\] (4.21)

By repeating the procedure, \(t^*\) is extended to \(T\), hence (4.21) holds for all \(t \geq 0\). Thus
\[
J(t) \geq J(0) + \left[ J'(0)^2 - 4\gamma^2 E(0)\phi^{-2\gamma-1}(0) \right]^{1/2} t, \quad \forall t > 0.
\] (4.22)

We choose \(T_0\) such that
\[
\frac{-J(0)}{\left[ J'(0)^2 - 4\gamma^2 E(0)\phi^{-2\gamma-1}(0) \right]^{1/2}} \leq T_0.
\]

Then there exists a positive number \(t_3\), such that \(J(t_3) = 0\) and \(0 < t_3 \leq T^* = \frac{-J(0)}{\left[ J'(0)^2 - 4\gamma^2 E(0)\phi^{-2\gamma-1}(0) \right]^{1/2}}\).

Thus, \(\phi(t) \to \infty\) as \(t \to t_3\). This is a contradiction with the fact that the maximal time of the existence of the solution is infinite. The theorem is proved.

**Example 4.1.** For the Cauchy problem (1.9)–(1.10), we take specific functions \(g(s), u_0(x)\) and \(u_1(x)\) satisfying the conditions of Theorem 2.3. We first discuss:

(I) The case \(E(0) < 0\). To this end, we take \(u_0(x) = u_1(x) = e^{-x^2}\) and \(g(s) = k_2 s^3\), obviously, \(u_0(x) \in H^2(\mathbb{R}), u_1(x) \in L^2(\mathbb{R}), g(0) = 0, G(u_0) \in L^1(\mathbb{R}), sg(s) = k_3 s^4\) and \((1 + 2\gamma)G(s) = \frac{1}{2}(1 + 2\gamma)k_2 s^4\). Thus when \(\gamma = \frac{1}{2}, g(s)\) satisfies assumption (2.1) of the Theorem 2.3. After some simple calculation, it implies \(||u_0||^2 = ||u_1||^2 = \sqrt{\frac{\pi}{2}}, ||u_{0,xx}||^2 = 3 \sqrt{\frac{\pi}{2}}, \int_{\mathbb{R}} G(u_0)dx = \frac{3\sqrt{\pi}}{32}\), we choose \(k_2 = -100\), then
\[
E(0) = ||u_1||^2 + ||u_{0,xx}||^2 + 2 \int_{\mathbb{R}} G(u_0)dx = 4 \sqrt{\frac{\pi}{2}} - 600 \sqrt{\frac{\pi}{2}}.
\] (4.23)

Hence, we see from (4.23) that \(E(0) < 0\), then the conditions of Theorem 2.3 are satisfied. Hence there exists a \(t_1 \leq \frac{600}{72\sqrt{\pi}}\) such that \(\phi(t) \to \infty\) as \(t \to t_1\).

(II) The case \(E(0) = 0\). We take \(u_0(x) = e^{-x^2}, u_1(x) = 5e^{-x^2}\) and \(g(s) = k_3 s^3\), then \(u_0(x) \in H^2(\mathbb{R}), u_1(x) \in L^2(\mathbb{R}), G(u_0) \in L^1(\mathbb{R}), \gamma = \frac{1}{2}, g(s)\) satisfies assumption (2.1) of the Theorem 2.3, and from above relations that \(||u_0||^2 = \sqrt{\frac{\pi}{2}}, ||u_1||^2 = 25 \sqrt{\frac{\pi}{2}}, ||u_{0,xx}||^2 = 3 \sqrt{\frac{\pi}{2}}, \int_{\mathbb{R}} G(u_0)dx = \frac{3\sqrt{\pi}}{32}\) and when \(k_1 = 1, k_3 = -\frac{448}{3\sqrt{\pi}}\), we obtain
\[
E(0) = ||u_1||^2 + ||u_{0,xx}||^2 + 2 \int_{\mathbb{R}} G(u_0)dx = 25 \sqrt{\frac{\pi}{2}} + 3 \sqrt{\frac{\pi}{2}} - 28 \sqrt{\frac{\pi}{2}} = 0.
\] (4.24)
\[
2(u_0, u_1) - \frac{1}{\gamma}(\|u_{01x}\|^2 + k_1\|u_{01}\|^3) = 2\sqrt{\frac{\pi}{2}} > 0. \tag{4.25}
\]

Then the conditions of Theorem 2.3 are satisfied. Hence there exists a \( t_2 \leq \frac{\phi(0)}{\gamma \psi(0)} \) such that \( \phi(t) \to \infty \) as \( t \to t_2 \).

(III) The case \( E(0) > 0 \). Now we take \( u_0(x) = e^{-x^2}, u_1(x) = e^{-x^2} \) and \( g(s) = s^2 \), then \( u_0(x) \in H^2(\mathbb{R}), u_1(x) \in L^2(\mathbb{R}), G(u_{0x}) \in L^1(\mathbb{R}) \), when \( \gamma = \frac{1}{2} \), \( g(s) \) satisfies assumption (2.1) of the Theorem 2.3. We can obtain \( \|u_0\|^2 = \|u_1\|^2 = \sqrt{\frac{\pi}{2}}, \|u_{01x}\|^2 = 3\sqrt{\frac{\pi}{2}}, \int_{\mathbb{R}} u_0 u_1 dx = \sqrt{\frac{\pi}{2}}, \)

\[
E(0) = \|u_1\|^2 + \|u_{01x}\|^2 + 2\int_{\mathbb{R}} G(u_{0x}) dx = 4\sqrt{\frac{\pi}{2}} - 2 \cdot \frac{8}{3} \int_{\mathbb{R}} x^3 e^{-3x^2} dx = 4\sqrt{\frac{\pi}{2}} > 0. \tag{4.26}
\]

Taking \( T_0 \) small enough such that

\[
(u_0, u_1) > [E(0)(\|u_0\|^2 + T_0\|u_{01x}\|^2 + T_0\|u_{01}\|^3)]^{\frac{1}{2}}.
\]

Then the conditions of Theorem 2.3 are satisfied. Hence there exists a finite positive number \( t_3 \), such that \( J(t_3) = 0 \) and \( 0 < t_3 \leq T^* = \frac{J(0)}{[J(0)^2 - 4\sqrt{2} E(0) \phi(0) \psi(0)]^{\frac{1}{2}}} \). Thus, \( \phi(t) \to \infty \) as \( t \to t_3 \).

5. Discussion

It is well known that Equation (1.11) describes the motion of the elastomer rod with internal damping. In the process of high speed movement, by the impact on damping characteristic and external excitation, the state of the elastomer rod is complicated and unpredictable at the initial velocity. Considering this situation, we choose initial data belonging to more general functional space \( u_0 \in W^{1,1}(\mathbb{R}) \cap H^1(\mathbb{R}), u_1 \in L^1(\mathbb{R}) \cap H^{-1}(\mathbb{R}) \). By using the \( L_1 \)-based spaces instead of \( L_2 \)-based ones, which are completely different from those used in [11], [18], we can still obtain the global smooth solution in the generalized space. In this paper, we just consider the problems in 1-dimensional space, but in high-dimensional space, do both Equation (1.9) and Equation (1.11) have and global solutions to the Cauchy problem or the initial boundary value problem? The question is interesting and opening.

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Conflict of Interest

The authors declare that there is no conflicts of interest in this paper.
References


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