Research article

$L^p$-analysis of one-dimensional repulsive Hamiltonian with a class of perturbations

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Abstract: The spectrum of one-dimensional repulsive Hamiltonian with a class of perturbations $H_p = -\frac{d^2}{dx^2} - x^2 + V(x)$ in $L^p(\mathbb{R})$ $(1 < p < \infty)$ is explicitly given. It is also proved that the domain of $H_p$ is embedded into weighted $L^q$-spaces for some $q > p$. Additionally, non-existence of related Schrödinger ($C_0$)-semigroup in $L^p(\mathbb{R})$ is shown when $V(x) \equiv 0$.

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1. Introduction

In this paper we consider

$$H := -\frac{d^2}{dx^2} - x^2 + V(x) \quad (1)$$

in $L^p(\mathbb{R})$, where $V \in C(\mathbb{R})$ is a real-valued and satisfies $V(x) \geq -a(1 + x^2)$ for some constant $a \geq 0$ and

$$\int_{\mathbb{R}} \frac{|V(x)|}{\sqrt{1 + x^2}} \, dx < \infty. \quad (2)$$

The operator (1) describes the quantum particle affected by a strong repulsive force from the origin. In fact, in the classical sense the corresponding Hamiltonian (functional) is given by $\hat{H}(x, p) = p^2 - x^2$ and then the particle satisfying $\dot{x} = \partial_p \hat{H}$ and $\dot{p} = -\partial_x \hat{H}$ goes away much faster than that for the free Hamiltonian $\hat{H}_0(x, p) = p^2$.

In the case where $p = 2$, the essential selfadjointness of $H$, endowed with the domain $C_0^\infty(\Omega)$, has been discussed by Ikebe and Kato [7]. After that several properties of $H$ is found out in a mount of
subsequent papers (for studies of scattering theory e.g., Bony et al. [2], Nicoleau [10] and also Ishida [8]).

In contrast, if $p$ is different from 2, then the situation becomes complicated. Actually, papers which deals with the properties of $H$ is quite few because of absence of good properties like symmetricity. In the $L^p$-framework, it is quite useful to consider the accretivity and sectoriality of the second-order differential operators. In fact, the case $-\frac{d^2}{dx^2} + V(x)$ with a nonnegative potential $V$ is formally sectorial in $L^p$, and therefore one can find many literature even $N$-dimensional case (e.g., Kato [9], Goldstein [6], Tanabe [14], Engel-Nagel [5]). However, it seems quite difficult to describe such a kind of non-accretive operators in a certain unified theory in the literature.

The present paper is in a primary position to make a contribution for theory of non-accretive operators in $L^p$ as mentioned above. The aim of this paper is to give a spectral properties of $H = -\frac{d^2}{dx^2} - x^2 + V(x)$ for the case where $V(x)$ can be regarded as a perturbation of the leading part $-\frac{d^2}{dx^2} - x^2$; note that if $V(x) = [\log(e + |x|)]^{-\alpha}$ ($\alpha \in \mathbb{R}$), then $\alpha < 1$ is admissible, which is same threshold as in the short range potential for $-\frac{d^2}{dx^2} - x^2$ stated in Bony [2] and also Ishida [8].

Here we define the minimal realization $H_{p,\min}$ of $H$ in $L^p = L^p(\mathbb{R})$ as

\[
\begin{aligned}
D(H_{p,\min}) := C_0^\infty(\mathbb{R}), \\
H_{p,\min}u(x) := -u''(x) - x^2u(x) + V(x)u(x).
\end{aligned}
\]

**Theorem 1.1.** For every $1 < p < \infty$, $H_{p,\min}$ is closable and the spectrum of the closure $H_p$ is explicitly given as

\[
\sigma(H_p) = \left\{ \lambda \in \mathbb{C} ; |\text{Im } \lambda| \leq \left| 1 - \frac{2}{p} \right| \right\}.
\]

Moreover, for every $1 < p < q < \infty$, one has consistence of the resolvent operators:

\[
(\lambda + H_p)^{-1}f = (\lambda + H_q)^{-1}f \text{ a.e. on } \mathbb{R} \quad \forall \lambda \in \rho(H_p) \cap \rho(H_q), \quad \forall f \in L^p \cap L^q.
\]

**Remark 1.1.** If $p = 2$, then our assertion is nothing new. The crucial part is the case $p \neq 2$ which is the case where the symmetricity of $H$ breaks down. The similar consideration for $-\frac{d^2}{dx^2} + V$ (but in $L^2$-setting) can be found in Dollard-Friedman [4].

This paper is organized follows: In Section 2, we prepare two preliminary results. In Section 3, we consider the fundamental systems of $\lambda u + Hu = 0$, and estimate the behavior of their solutions. By virtue of that estimates, we will describe the resolvent set of $H_p$ in Section 4. In section 5, we prove never to be generated $C_0$-semigroups by $\pm iH_p$ under the condition $V = 0$.

## 2. Preliminary results

First we state well-known results for the essentially selfadjointness of Schrödinger operators in $L^2$ which is firstly described in [7]. We would like to refer also Okazawa [12].

**Theorem 2.1** (Okazawa [12, Corollary 6.11]). Let $V(x)$ be locally in $L^2(\mathbb{R})$ and assume that $V(x) \geq -c_1 - c_2|x|^2$, where $c_1, c_2 \geq 0$ are constants. Then $H_{2,\min}$ is essentially selfadjoint.
Next we note the asymptotic behavior of solutions to second-order linear ordinary differential equations of the form

\[ y''(x) = (\Phi(x) + \Psi(x))y(x) \]

in which the term \( \Psi(x)y(x) \) can be treated as a perturbation of the leading part \( \Phi(x)y(x) \).

**Theorem 2.2** (Olver [13, Theorem 6.2.2 (p.196)]). In a given finite or infinite interval \((a_1, a_2)\), let \( a \in (a_1, a_2) \), \( \Psi(x) \) a positive, real, twice continuously differentiable function, \( \Psi(x) \) a continuous real or complex function, and

\[
F(x) = \int \left\{ \frac{1}{\Phi(x)^{1/4}} \frac{d^2}{dx^2} \left( \frac{1}{\Phi(x)^{1/4}} \right) - \frac{\Psi(x)}{\Phi(x)^{1/2}} \right\} dx.
\]

Then in this interval the differential equation

\[
\frac{d^2w}{dx^2} = (\Phi(x) + \Psi(x))w
\]

has twice continuously differential solutions

\[
w_1(x) = \frac{1}{\Phi(x)^{1/4}} \exp \left\{ i \int \Phi(x)^{1/2} \, dx \right\} (1 + \varepsilon_1(x)),
\]

\[
w_2(x) = \frac{1}{\Phi(x)^{1/4}} \exp \left\{ -i \int \Phi(x)^{1/2} \, dx \right\} (1 + \varepsilon_2(x)),
\]

such that

\[ |\varepsilon_j(x)|, \frac{1}{\Phi(x)^{1/2}}|\varepsilon_j(x)| \leq \exp \left\{ \frac{1}{2} V_{a_j,a}(F) \right\} - 1 \quad (j = 1, 2) \]

provided that \( V_{a_1,a}(F) < \infty \) (where \( V_{a_1,a}(F) = \int |F'(t)| \, dt \) is the total variation of \( F \)). If \( \Psi(x) \) is real, then the solutions \( w_1(x) \) and \( w_2(x) \) are complex conjugates.

For the above theorem, see also Beals-Wong [1, 10.12, p.355].

**3. Fundamental systems of** \( \lambda u - u'' - x^2 u + Vu = 0 \)

**3.1. The case** \( \lambda \in \mathbb{R} \)

We consider the behavior of solutions to

\[ \lambda u(x) - u''(x) - x^2 u(x) + V(x)u(x) = 0, \quad x \in \mathbb{R}, \]  \hspace{1cm} (4)

where \( \lambda \in \mathbb{R} \).

**Proposition 3.1.** There exist solutions \( u_{a,1}, u_{a,2} \) of (4) such that \( u_{a,1} \) and \( u_{a,2} \) are linearly independent and satisfy

\[ |u_{a,1}(x)| \leq C_a(1 + |x|)^{-\frac{1}{4}}, \quad |u_{a,2}(x)| \leq C_a(1 + |x|)^{-\frac{1}{4}} \quad \forall x \in \mathbb{R}, \]

\[ |u_{a,1}(x)| \geq \frac{1}{2}(1 + |x|)^{-\frac{1}{4}}, \quad |u_{a,2}(x)| \geq \frac{1}{2}(1 + |x|)^{-\frac{1}{4}} \quad \forall x \geq R_a \]

for some constants \( C_a, R_a > 0 \) independent of \( x \). In particular, \( u_{a,1}, u_{a,2} \in L^p(\mathbb{R}) \) if and only if \( 2 < p < \infty \).
Proof. First we consider (4) for \( x > 0 \). Using the Liouville transform

\[
v(y) := (2y)^{-\frac{1}{2}} u[(2y)^{\frac{1}{2}}], \quad \text{or equivalently,} \quad u(x) = x^{-\frac{1}{2}} v \left( \frac{x^2}{2} \right),
\]

we have

\[
(\lambda - x^2) x^{-\frac{1}{2}} v \left( \frac{x^2}{2} \right) = u''(x) - V(x)u(x) = x^{\frac{3}{2}} v'' \left( \frac{x^2}{2} \right) + \frac{3}{4} x^{-\frac{1}{2}} v \left( \frac{x^2}{2} \right) - x^{-\frac{3}{2}} V(x) v \left( \frac{x^2}{2} \right).
\]

Therefore noting that \( y = x^2 / 2 \), we see that

\[
v''(y) = \left[ -\left(1 - \frac{\lambda}{4y} \right)^2 + \frac{\lambda^2 - 3}{16y^2} + \frac{V((2y)^{\frac{1}{2}})}{2y} \right] v(y) = (\Phi(y) + \Psi(y)) v(y). \tag{5}
\]

Here we have put for \( y > 0 \),

\[
\Phi(y) := -\left(1 - \frac{\lambda}{4y} \right)^2, \quad \Psi(y) := \frac{\lambda^2 - 3}{16y^2} + \frac{V((2y)^{\frac{1}{2}})}{2y}.
\]

Let

\[
\Pi(y) := |\Phi(y)|^{-\frac{1}{2}} \left(-\frac{d^2}{dx^2} + \Psi(y)\right)|\Phi(y)|^{-\frac{1}{2}}, \quad y \geq \lambda_+ := \max(\lambda, 0).
\]

Then we see that for every \( y \geq \lambda_+ \),

\[
|\Pi(y)| \leq \left(1 - \frac{\lambda}{4y}\right)^{-3} \frac{3\lambda^2}{64y^2} + \left(1 - \frac{\lambda}{4y}\right)^{-2} \frac{\lambda}{4y^3} + \left(1 - \frac{\lambda}{4y}\right)^{-1} \frac{|\lambda^2 - 3|}{16y^2} + \frac{|V((2y)^{\frac{1}{2}})|}{2y} \leq \frac{M_{\lambda}}{y^2} + \frac{|V((2y)^{\frac{1}{2}})|}{2y},
\]

where \( M_{\lambda} \) is a positive constant depending only on \( \lambda \). Therefore

\[
\int_{\lambda_+}^{\infty} |\Pi(y)| \, dy \leq M_{\lambda} \int_{\lambda_+}^{\infty} \frac{1}{y^2} \, dy + \int_{\lambda_+}^{\infty} \frac{|V(x)|}{x} \, dx < \infty.
\]

Thus \( \Pi \in L^1((\lambda_+, \infty)) \). By Theorem 2.2, we obtain that there exists a fundamental system \((v_{\lambda,1,1}, v_{\lambda,1,2})\) of (5) such that

\[
v_{\lambda,1,1}(y) y^{\frac{1}{4}} e^{-iy} \to 1, \quad v_{\lambda,1,2}(y) y^{-\frac{3}{4}} e^{iy} \to 1 \quad \text{as} \quad y \to \infty
\]

(see also [11]). Taking \( u_{\lambda,j}(x) = x^{-\frac{1}{2}} v_{\lambda,j}(x^2 / 2) \) for \( j = 1, 2 \), we obtain that \((u_{\lambda,1}, u_{\lambda,2})\) is a fundamental system of (4) on \((\lambda_+, \infty)\) and

\[
u_{\lambda,1,1}(y) x^{\frac{1}{4} + iy} e^{-\frac{1}{2} x} \to 2^{-\frac{1}{4} x}, \quad u_{\lambda,1,2}(x) x^{\frac{1}{4} - iy} e^{\frac{1}{2} x} \to 2^{\frac{1}{4} x},
\]

as \( x \to \infty \). The above fact implies that there exists a constant \( R_{\lambda} > \lambda_+ \) such that

\[
\frac{1}{2} x^{-\frac{3}{2}} \leq |u_{\lambda,j}(x)| \leq \frac{3}{2} x^{-\frac{3}{2}}, \quad x \geq R_{\lambda}, \quad j = 1, 2.
\]

We can extend \((u_{\lambda,1}, u_{\lambda,2})\) as a fundamental system on \( \mathbb{R} \). By applying the same argument as above to (4) for \( x < 0 \), we can construct a different fundamental system \((\tilde{u}_{\lambda,1}, \tilde{u}_{\lambda,2})\) on \( \mathbb{R} \) satisfying

\[
\frac{1}{2} |x|^{-\frac{3}{2}} \leq |\tilde{u}_{\lambda,j}(x)| \leq \frac{3}{2} |x|^{-\frac{3}{2}}, \quad x \leq -R_{\lambda}, \quad j = 1, 2.
\]

By definition of fundamental system, \( u_{\lambda,j} \) can be rewritten as

\[
u_{\lambda,1,1}(x) = c_{11} \tilde{u}_{\lambda,1,1}(x) + c_{12} \tilde{u}_{\lambda,1,2}(x), \quad u_{\lambda,2}(x) = c_{21} \tilde{u}_{\lambda,1,1}(x) + c_{22} \tilde{u}_{\lambda,2,2}(x).
\]

Hence we have the upper and lower estimates of \( u_{\lambda,j} \) (\( j = 1, 2 \)), respectively. \( \square \)
3.2. The case $\lambda \in \mathbb{C} \setminus \mathbb{R}$

We consider the behavior of solutions to

$$\lambda u(x) - u''(x) - x^2u(x) + V(x)u(x) = 0,$$  \hspace{1cm} (6)

where $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with $\text{Im} \lambda > 0$. The case $\text{Im} \lambda < 0$ can be reduced to the problem $\text{Im} \lambda > 0$ via complex conjugation.

3.2.1. Properties of solutions to an auxiliary problem

We start with the following function $\varphi_\lambda$:

$$\varphi_\lambda(x) := x^{-\frac{1+\lambda}{2}} e^{\frac{i\lambda}{2}}, \hspace{1cm} x > 0.$$  \hspace{1cm} (7)

Then by a direct computation we have

Lemma 3.2. $\varphi_\lambda$ satisfies

$$\lambda \varphi_\lambda - \varphi_\lambda'' - x^2\varphi_\lambda + g_\lambda \varphi_\lambda = 0, \hspace{1cm} x \in (0, \infty),$$  \hspace{1cm} (8)

where $g_\lambda(x) := \frac{(1+i)(3+3i)}{4x^2}, \hspace{1cm} x > 0$.

Remark 3.1. If $\lambda = i$ or $\lambda = 3i$, then $\varphi_\lambda$ is nothing but a solution of the original equation (6) with $V = 0$.

Next we construct another solution of (8) which is linearly independent of $\varphi_\lambda$. Before construction, we prepare the following lemma.

Lemma 3.3. Let $\lambda$ satisfy $\text{Im} \lambda > 0$ and let $\varphi_\lambda$ be given in (7). Then for every $a > 0$, there exists $F_a \in \mathbb{C}$ such that

$$\int_a^x \varphi_\lambda(t)^{-2} \, dt \rightarrow F_a \hspace{1cm} \text{as} \hspace{1cm} x \rightarrow \infty$$

and then $x \mapsto \int_a^x \varphi_\lambda(t)^{-2} \, dt - F_a$ is independent of $a$. Moreover, for every $x > 0$,

$$\left| \int_a^x \varphi_\lambda(t)^{-2} \, dt - F_a - i \frac{\lambda}{2} x^{\text{Re} \lambda} e^{-ix^2} \right| \leq C_a x^{-\text{Im} \lambda - 2},$$

where $C_a := \frac{|\lambda|}{4} \left( 1 + \sqrt{1 + \left( \frac{\text{Re} \lambda}{\text{Im} \lambda + 2} \right)^2} \right)$.

Remark 3.2. If $a = 0$ and $\lambda = i$, then $F_0$ gives the Fresnel integral $\lim_{x \to \infty} \int_0^x e^{-it^2} \, dt$. Hence $F_0 = \sqrt{\pi/8}(1 - i)$.

Proof. By integration by part, we have

$$\int_a^x t^{1+\lambda i} e^{-it^2} \, dt = \left( \frac{i}{2} x^{\lambda i} e^{-ix^2} - \frac{i}{2} a^{\lambda i} e^{-ia^2} \right) + \frac{\lambda i}{4} \left( x^{\lambda i - 2} e^{-ix^2} - a^{\lambda i - 2} e^{-ia^2} \right) - \frac{\lambda i(\lambda i - 2)}{4} \int_a^x t^{\lambda i - 3} e^{-it^2} \, dt.$$

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Noting that \( t^{i \lambda - 3} e^{-it^2} \) is integrable in \((a, \infty)\), we have
\[
\int_a^x t^{i \lambda i} e^{-it^2} dt \to - \frac{i}{2} a^{i \lambda} e^{-ia^2} - \frac{\lambda i (\lambda i - 2)}{4} \int_a^x t^{i \lambda - 3} e^{-it^2} dt =: F^A_a
\]
as \( x \to \infty \). And therefore \( \int_a^x t^{i \lambda i} e^{-it^2} dt - F^A_a \) is independent of \( a \) and
\[
\left| \int_a^x t^{i \lambda i} e^{-it^2} dt - F^A_a - \frac{i}{2} x^{i \lambda} e^{-ix^2} \right| = \left| \frac{\lambda}{4} x^{i \lambda - 2} e^{-ix^2} + \frac{\lambda i (\lambda i - 2)}{4} \int_a^x t^{i \lambda - 3} e^{-it^2} dt \right| \leq C_A x^{i \lambda - 2}.
\]
This is nothing but the desired inequality. \( \square \)

**Lemma 3.4.** Let \( \varphi_\lambda \) be as in (7) and define \( \psi_\lambda \) as
\[
\psi_\lambda(x) := \varphi_\lambda(x) \int_a^x \frac{1}{\varphi_\lambda(t)^2} dt - F^A_a \varphi_\lambda(x), \quad x > 0.
\]
Then \( \psi_\lambda \) is independent of \( a \) and \((\varphi_\lambda, \psi_\lambda)\) is a fundamental system of (8). Moreover, there exists \( a_0 > 0 \) such that
\[
\frac{1}{3} x^{\frac{i \lambda + i + 1}{2}} \leq |\psi_\lambda(x)| \leq x^{\frac{i \lambda + i + 1}{2}}, \quad x \in [a_0, \infty).
\]

**Proof.** From Lemma 3.3 we have
\[
x^{\frac{i \lambda + i + 1}{2}} \left| \psi_\lambda(x) - \frac{i}{2} x^{\frac{i \lambda - 1}{2}} e^{-ix^2} \right| = x^{\frac{i \lambda + i + 1}{2}} |\varphi_\lambda(x)| \left| \int_a^x \frac{1}{\varphi_\lambda(t)^2} dt - F^A_a - \frac{i}{2} t^{i \lambda} e^{-it^2} \right| \leq C_A x^{-2}.
\]
Putting \( a_0 = (6C_A)^{\frac{1}{3}} \), we deduce the desired assertion. \( \square \)

### 3.2.2. Fundamental system of the original problem

Next we consider
\[
\lambda w - w'' - x^2 w + g_\lambda w = \tilde{g}_\lambda h, \quad x > 0
\]
with a given function \( h \), where \( g_\lambda \) is given as in Lemma 3.2 and \( \tilde{g}_\lambda := g_\lambda - V \). To construct solutions of (6), we will define two types of solution maps \( h \mapsto w \) and consider their fixed points.

First we construct a solution of (6) which behaves like \( \psi_\lambda \) at infinity.

**Definition 3.5.** For \( b > 0 \), define
\[
U_b h(x) := \psi_\lambda(x) - \psi_\lambda(x) \int_b^x \varphi_\lambda(s) \tilde{g}_\lambda(s) h(s) ds - \psi_\lambda(x) \int_x^\infty \varphi_\lambda(s) \tilde{g}_\lambda(s) h(s) ds, \quad x \in [b, \infty)
\]
for \( h \) belonging to a Banach space
\[
X_\lambda(b) := \left\{ h \in C([b, \infty)) : \sup_{x \in [b, \infty)} \left( x^{\frac{i \lambda + i + 1}{2}} |h(x)| \right) < \infty \right\}, \quad \|h\|_{X_\lambda(b)} := \sup_{x \in [b, \infty)} \left( x^{\frac{i \lambda + i + 1}{2}} |h(x)| \right).
\]
Lemma 3.6. In Definition 3.5 we deal with such a solution with particular, for arbitrary fixed $x$.

Proof. Let $h$ be such an arbitrary fixed $x$. Then $w_{c_1,c_2} \in C([b, \infty))$. In particular, for $x \geq b_2$,

$$w_{c_1,c_2}(x) = c_1 \varphi_4(x) + c_2 \psi_4(x) + \int_b^x \left( \varphi_4(x) \psi_4(s) - \varphi_4(s) \psi_4(x) \right) \tilde{g}_A(s) h(s) \, ds,$$

where $c_1, c_2 \in \mathbb{C}$. Suppose that $h \in C_0^\infty((b, \infty))$ with supp $h \subset [b_1, b_2]$. Then $w_{c_1,c_2} \in C([b, \infty))$. In particular, for $x \geq b_2$,

$$w_{c_1,c_2}(x) = \left( c_1 + \int_{b_1}^{b_2} \psi_4(s) \tilde{g}_A(s) h(s) \, ds \right) \varphi_4(x) + \left( c_2 - \int_{b_1}^{b_2} \psi_4(s) \tilde{g}_A(s) h(s) \, ds \right) \psi_4(x).$$

Therefore $w_{c_1,c_2}$ behaves like $\psi_4$ (that is, $w_{c_1,c_2} \in X_\delta(b)$) only when

$$c_1 = - \int_{b_1}^{b_2} \psi_4(s) \tilde{g}_A(s) h(s) \, ds = - \int_b^{\infty} \psi_4(s) \tilde{g}_A(s) h(s) \, ds.$$

In Definition 3.5 we deal with such a solution with $c_2 = 1$.

Well-definedness of $U$ in Definition 3.5 and its contractivity are proved in next lemma.

**Lemma 3.6.** The following assertions hold:

1. For every $b > 0$, the map $U : X_\delta(b) \to X_\delta(b)$ is well-defined;
2. There exists $b \delta > 0$ such that $U$ is contractive in $X_\delta(b)$ with

$$\|Uh_1 - Uh_2\|_{X_\delta(b)} \leq \frac{1}{5} \|h_1 - h_2\|_{X_\delta(b)}, \quad h_1, h_2 \in X_\delta(b)$$

and then $U$ has a unique fixed point $w_1 \in X_\delta(b)$;
3. $w_1$ can be extended to a solution of (6) in $\mathbb{R}$ satisfying

$$\frac{1}{12} x^{-\frac{\max{1, 1}}{2}} \leq |w_1(x)| \leq 2 x^{-\frac{\max{1, 1}}{2}}, \quad x \in [b_\delta, \infty).$$

**Proof.** (i) By Lemma 3.4 we have $\psi_4 \in X_\delta(b)$. Therefore to prove well-definedness of $U$, it suffices to show that the second term in the definition of $U$ belongs to $X_\delta(b)$.

Let $h \in X_\delta(b)$. Then for $x \in [b, \infty)$,

$$x^{-\frac{\max{1, 1}}{2}} \left| \varphi_4(x) \int_b^x \psi_4(s) \tilde{g}_A(s) h(s) \, ds \right| \leq x^{-\frac{\max{1, 1}}{2}} \|\varphi_4\|_{L^1} \int_b^x s^{-1/2} |\tilde{g}_A(s)| |h(s)| \, ds \leq \|\varphi_4\|_{L^1} \|s^{-1/2} \tilde{g}_A\|_{L^1(b, \infty)}$$

and

$$x^{-\frac{\max{1, 1}}{2}} \left| \psi_4(x) \int_b^x \varphi_4(s) \tilde{g}_A(s) h(s) \, ds \right| \leq \|\psi_4\|_{L^1} \int_b^x s^{-1} |\tilde{g}_A(s)| |h(s)| \, ds \leq \|\psi_4\|_{L^1} \|s^{-1} \tilde{g}_A\|_{L^1(b, \infty)}.$$

Hence we have $Uh \in C((b, \infty))$ and therefore $Uh \in X_\delta(b)$, that is, $U : X_\delta(b) \to X_\delta(b)$ is well-defined.

(ii) Let $h_1, h_2 \in X_\delta(b)$. Then we have

$$Uh_1(x) - Uh_2(x) = -\psi_4(x) \int_b^x \varphi_4(s) \tilde{g}_A(s)(h_1(s) - h_2(s)) \, ds - \varphi_4(x) \int_b^x \psi_4(s) \tilde{g}_A(s)(h_1(s) - h_2(s)) \, ds.$$
Proceeding the same computation as above, we deduce
\[ \|Uh_1 - Uh_2\|_{X(b)} \leq 2\|s^{-1}g_\alpha\|_{L^1(b,\infty)}\|h_1 - h_2\|_{X(b)}. \]

Choosing \( b \) large enough, we obtain \( \|Uh_1 - Uh_2\|_{X(b)} \leq 5^{-1}\|h_1 - h_2\|_{X(b)} \), that is \( U \) is contractive in \( X(b) \). By contraction mapping principle, we obtain that \( U \) has a unique fixed point \( w_1 \in X(b) \).

(iii) Since \( w_1 \) satisfies (10) with \( h = w_1 \), \( w_1 \) is a solution of the original equation (6) in \([b, \infty)\). As in the last part of the proof of Proposition 3.1, we can extend \( w_1 \) as a solution of (6) in \( \mathbb{R} \). Since \( Uw_1 = w_1 \) and \( U0 = \psi_\lambda \), it follows from the contractivity of \( U \) that
\[ \|w_1 - \psi_\lambda\|_X = \|Uw_1 - U0\|_X \leq \frac{1}{5}\|w_1\|_X \leq \frac{1}{5}\|w_1 - \psi_\lambda\|_X + \frac{1}{5}\|\psi_\lambda\|_X. \]

Consequently, we have \( \|w_1 - \psi_\lambda\|_X \leq 4^{-1}\|\psi_\lambda\|_X \leq 4^{-1} \) and then for \( x \geq b \),
\[ |w_1(x)| \geq |\psi_\lambda(x)| - |w_1(x) - \psi_\lambda(x)| \geq \left( \frac{1}{3} - \|w_1 - \psi_\lambda\|_X \right) x^{-\frac{\ln 4}{3}} \geq \frac{1}{12} x^{-\frac{\ln 4}{3}}. \]

Next we construct another solution of (6) which behaves like \( \varphi_\lambda \) at infinity.

**Definition 3.7.** Let \( b > 0 \) be large enough. Define
\[ \bar{U}h(x) := \varphi_\lambda(x) + \int_b^x (\varphi_\lambda(s)\psi_\lambda(s) - \varphi_\lambda(s)\varphi_\lambda(x))\bar{g}_\lambda(s)h(s) \, ds \]
for \( h \) belonging to a Banach space
\[ Y_\lambda(b) := \left\{ h \in C([b, \infty)) : \sup_{x \in [b, \infty)} \left( x^{-\frac{\ln 4}{3}} |h(x)| \right) < \infty \right\}, \quad \|h\|_{Y_\lambda(b)} := \sup_{x \in [b, \infty)} \left( x^{-\frac{\ln 4}{3}} |h(x)| \right). \]

**Lemma 3.8.** The following assertions hold:

(i) for every \( b > 0 \), the map \( \bar{U} : Y_\lambda(b) \to Y_\lambda(b) \) is well-defined;

(ii) there exists \( b_\lambda > 0 \) such that \( \bar{U} \) is contractive in \( Y_\lambda(b_\lambda) \) with
\[ \|\bar{U}h_1 - \bar{U}h_2\|_{Y_\lambda(b_\lambda)} \leq \frac{1}{5}\|h_1 - h_2\|_{Y_\lambda(b_\lambda)}, \quad h_1, h_2 \in Y_\lambda(b_\lambda) \]
and then \( \bar{U} \) has a unique fixed point \( \bar{w}_1 \in Y_\lambda(b_\lambda) \);

(iii) \( \bar{w}_1 \) can be extended to a solution of (6) in \( \mathbb{R} \) satisfying
\[ \frac{1}{2} x^{-\frac{\ln 4}{3}} \leq |\bar{w}_1(x)| \leq 2x^{-\frac{\ln 4}{3}}, \quad x \in [b_\lambda, \infty). \]

**Proof.** The proof is similar to the one of Lemma 3.6. \( \square \)
Considering the equation (6) for \( x < 0 \), we also obtain the following lemma.

**Lemma 3.9.** For every \( \lambda \in \mathbb{C} \) with \( \text{Im} \lambda > 0 \), there exist a fundamental system \( (w_1, w_2) \) of (6) and positive constants \( c_1, C_A, R_A \) such that

\[
|w_1(x)| \leq C_A(1 + |x|)^{\frac{\text{Im} + \lambda}{2}}, \quad x \leq 0, \quad |w_1(x)| \leq C_A(1 + |x|)^{-\frac{\text{Im} + \lambda}{2}}, \quad x \geq 0, \quad (11)
\]

\[
|w_2(x)| \leq C_A(1 + |x|)^{\frac{\text{Im} + \lambda}{2}}, \quad x \leq 0, \quad |w_2(x)| \leq C_A(1 + |x|)^{-\frac{\text{Im} + \lambda}{2}}, \quad x \geq 0
\]

and

\[
|w_1(x)| \geq c_1(1 + |x|)^{-\frac{\text{Im} + \lambda}{2}}, \quad x \geq R_A, \quad |w_2(x)| \geq c_1(1 + |x|)^{-\frac{\text{Im} + \lambda}{2}}, \quad x \leq -R_A. \quad (13)
\]

**Proof.** In view of Lemma 3.6, it suffices to find \( w_2 \) satisfying the conditions above.

Let \( w_s \) and \( \tilde{w}_s \) be given as in Lemmas 3.6 and 3.8 with \( V(x) \) replaced with \( V(-x) \). Noting that \( w_1 \) can be rewritten as \( w_1(x) = c_1 w_s(-x) + c_2 \tilde{w}_s(-x) \), we see from Lemma 3.6 and 3.8 that (11) and the first half of (13) are satisfied. Set \( w_2(x) = w_s(-x) \) for \( x \in \mathbb{R} \). As in the same way, we can verify (12).

Finally, we prove the last half of (13). Since \( H_{2\text{min}} \) is essentially selfadjoint in \( L^2(\mathbb{R}) \), \( \lambda \) belongs to the resolvent set of \( H_2 \), that is, \( N(\lambda + H_2) = \{0\} \). This implies that \( w_2 \not\in L^2(\mathbb{R}) \). Noting that \( w_2 \in L^2((-\infty, 0)) \), we have \( w_2 \not\in L^2((0, \infty)) \). Now using the representation

\[
w_2(x) = c_1 w_1(x) + c_2 \tilde{w}_1(x), \quad x \in \mathbb{R},
\]

we deduce that \( c_2 \neq 0 \). Therefore using Lemma 3.6 (iii) and Lemma 3.8 (iii), we have

\[
|w_2(x)| \geq |c_2| |\tilde{w}_1(x)| - |c_1| |w_1(x)| \geq \frac{|c_2|}{2} x^{\frac{\text{Im} + \lambda}{2}} - 2|c_1| x^{-\frac{\text{Im} + \lambda}{2}} \geq \frac{|c_2|}{4} x^{-\frac{\text{Im} + \lambda}{2}}
\]

for \( x \) large enough. \( \square \)

### 4. Resolvent estimates in \( L^p \)

The following lemma, verified by the variation of parameters, gives a possibility of representation of the Green function for resolvent operator \( H \) in \( L^p \).

**Lemma 4.1.** Assume that \( \lambda \in \rho(\tilde{H}) \) in \( L^p \), where \( \tilde{H} \) is a realization of \( H \) in \( L^p \). Then for every \( u \in C_0^\infty(\mathbb{R}) \),

\[
u(x) = \frac{w_1(x)}{W_A} \int_{-\infty}^x w_2(s)f(s) \, ds + \frac{w_2(x)}{W_A} \int_x^\infty w_1(s)f(s) \, ds, \quad x \in \mathbb{R},
\]

where \( f := \lambda u - u'' - x^2 u + Vu \in C_0^\infty(\mathbb{R}) \) and \( W_A \neq 0 \) is the Wronskian of \( (w_1, w_2) \).

**Proposition 4.2.** Let \( 1 < p < \infty \). If \( |1 - \frac{2}{p}| < \text{Im} \lambda \), then the operator defined as

\[
R(\lambda)f(x) := \frac{w_1(x)}{W_A} \int_{-\infty}^x w_2(s)f(s) \, ds + \frac{w_2(x)}{W_A} \int_x^\infty w_1(s)f(s) \, ds, \quad f \in C_0^\infty(\mathbb{R})
\]

can be extended to a bounded operator on \( L^p \). More precisely, there exists \( M_A > 0 \) such that

\[
\|R(\lambda)f\|_{L^p} \leq M_A \left[ \|\text{Im} \lambda\|^2 - \left( 1 - \frac{2}{p} \right)^2 \right]^{-1} \|f\|_{L^p}, \quad f \in L^p(\mathbb{R}). \quad (14)
\]
In particular, $H_{p,\min}$ is closable and its closure $H_p$ satisfies

$$\left\{ \lambda \in \mathbb{C} : |\text{Im}\lambda| > \left| 1 - \frac{2}{p} \right| \right\} \subset \rho(H_p).$$

Proof. Let $f \in C_0^\infty(\mathbb{R})$. Set

$$u_1(x) := w_1(x) \int_{-\infty}^x w_2(s)f(s)\,ds, \quad u_2(x) := w_1(x) \int_x^\infty w_1(s)f(s)\,ds.$$ 

We divide the proof of $u_1 \in L^p(\mathbb{R})$ into two cases $x \geq 0$ and $x < 0$; since the proof of $u_2 \in L^p(\mathbb{R})$ is similar, this part is omitted.

The case $u_1$ for $x \geq 0$, it follows from Lemma 3.9 and Hölder inequality that

\[
|u_1(x)| \leq C_3^2 \left(1 + |x|\right)^{\frac{\Im\lambda + 1}{2} - \frac{p'}{2}} \left[ \int_{-\infty}^0 (1 + |s|)^{\frac{\Im\lambda + 1}{2} - \frac{p'}{2}} |f(s)|\,ds \right]^{\frac{1}{p'}} \\
\leq C_3^2 \left(\frac{\Im\lambda + 1}{2} - \frac{p'}{2} - 1\right)^{-\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}_+)} (1 + |x|)^{-\frac{p - 1}{p'}} \\
+ C_3^2 (1 + |x|)^{-\frac{\Im\lambda + 1}{2} - \frac{p'}{2}} \left( \int_0^x (1 + |s|)^{\frac{\Im\lambda + 1}{2} - \frac{p'}{2}} \,ds \right)^{\frac{1}{p'}} \left( \int_0^x (1 + |s|)^{\alpha p} |f(s)|^p \,ds \right)^{\frac{1}{p}}.
\]

with $0 < \alpha < \frac{\Im\lambda + 1}{2} + 1/p'$. By the triangle inequality we have

$$\|u_1\|_{L^p(\mathbb{R}_+)} \leq C_3^2 \left(\frac{\Im\lambda + 1}{2} - \frac{p'}{2} - 1\right)^{-\frac{1}{p'}} \left(\frac{\Im\lambda + 1}{2} - \frac{p'}{2} - 1\right)^{-\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}_+)} + I_1(\alpha)$$

and

\[
(I_1(\alpha))^p = C_4^2 \left(\frac{\Im\lambda - 1}{2} - \frac{p'}{2} - \alpha p' - 1\right)^{-\frac{1}{p'}} \frac{1}{(1 - \alpha p')^{-1}} \int_0^\infty (1 + |s|)^{-\alpha p} \left( \int_0^\infty (1 + |s|)^{\alpha p} |f(s)|^p \,ds \right) \,dx
\]

Choosing $\alpha = \frac{1}{p'} \left(\frac{\Im\lambda + 1}{2} - \frac{p'}{2} + 1\right)$, we obtain

$$\|u_1\|_{L^p(\mathbb{R}_+)} \leq C_3^2 \left(\frac{\Im\lambda + 1}{2} - \frac{p'}{2} - 1\right)^{-\frac{1}{p'}} \left(\frac{\Im\lambda + 1}{2} - \frac{p'}{2} - 1\right)^{-\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}_+)} + C_4^2 \left(\frac{\Im\lambda - 1}{2} + \frac{1}{p'} - 1\right)^{-1} \|f\|_{L^p(\mathbb{R}_+)}. $$

The case $u_1$ for $x < 0$, by the same way as the case $x > 0$, we have

$$|u_1(x)|^p \leq C_5^2 \left(\frac{\Im\lambda + 1}{2} - \frac{p'}{2} - \beta p' - 1\right)^{-\frac{1}{p'}} (1 + |x|)^{-\beta p} \int_{-\infty}^x (1 + |s|)^{-\beta p} |f(s)|^p \,ds, \quad (16)$$
where \(0 < \beta < \frac{1 + \beta}{p'} - \frac{1}{p'}\). Taking \(\beta = \frac{1}{p'} \left( \frac{1 + \beta}{p'} - 1 \right)\), we have

\[
\|u_1\|_{L^p(R_+)} \leq C_3^2 \left( \frac{\text{Im} \lambda + 1}{2} - \frac{1}{p'} \right)^{-1} \|f\|_{L^p(R_+)}.
\]

Proceeding the same argument for \(u_2\) and combining the estimates for \(u_1\) and \(u_2\), we obtain (14). \(\square\)

**Corollary 4.3.** Let \(\mathcal{R}(\lambda)\) be as in Proposition 4.2. Then for every \(f \in L^p(R_+), \mathcal{R}(\lambda)f \in C(R)\) and

\[
\sup_{x \in R} \left( (1 + |x|)^{\frac{1}{p}} |\mathcal{R}(\lambda)f(x)| \right) \leq \tilde{C}_4 \|f\|_{L^p}.
\]  

(17)

**Proof.** Let \(f \in C_0^\infty(R)\) and set \(u_1\) and \(u_2\) as in the proof of Proposition 4.2. Since the proof for \(u_1\) and \(u_2\) are similar, we only show the estimate of \(u_1\). From (15), we have for \(x \geq 0\),

\[
(1 + |x|)^{\frac{1}{p}} |u_1(x)| \leq C_4 \left( \frac{\text{Im} \lambda + 1}{2} p' - 1 \right)^{-\frac{1}{p'}} \|f\|_{L^p(R_+)} (1 + |x|)^{\frac{\text{Im} \lambda + 1}{2} - \frac{1}{p'}}
\]

\[
+ C_4^2 \left( \frac{\text{Im} \lambda - 1}{2} p' - \alpha p' + 1 \right)^{-\frac{1}{p'}} (1 + |x|)^{-\alpha} \left( \int_0^x (1 + |s|)^{\alpha p'} |f(s)|^p ds \right)^{\frac{1}{p'}}
\]

\[
\leq C_4 \left( \frac{\text{Im} \lambda + 1}{2} p' - 1 \right)^{-\frac{1}{p'}} \|f\|_{L^p(R_+)} + C_4^2 \left( \frac{\text{Im} \lambda - 1}{2} p' - \alpha p' + 1 \right)^{-\frac{1}{p'}} \|f\|_{L^p(R_+)},
\]

where \(0 < \alpha < \frac{1 + \beta}{2} + \frac{1}{p'}\). This implies (17) for \(x \geq 0\). If \(x \leq 0\), then from (16) we can obtain

\[
(1 + |x|)^{\frac{1}{p}} |u_1(x)| \leq C_4 \left( \frac{\text{Im} \lambda + 1}{2} p' - \beta p' - 1 \right)^{-\frac{1}{p'}} \|f\|_{L^p(R_+)},
\]

where \(0 < \beta < \frac{1 + \beta}{2} - \frac{1}{p'}\). This yields (17) for \(x \leq 0\). The proof is completed. \(\square\)

By interpolation inequality, we deduce the following assertion.

**Proposition 4.4.** Let \(1 < p < \infty\) and \(p \leq q \leq \infty\). Then

\[
D(H_p) \subset \left\{ w \in C(R) ; \langle x \rangle^{\frac{1}{p} - \frac{1}{q} + 1} w \in L^q \right\}.
\]

More precisely, there exists a constant \(C_{p,q} > 0\) such that

\[
\left\| \langle x \rangle^{\frac{1}{p} - \frac{1}{q} + 1} u \right\|_{L^q} \leq C_{p,q} \left( \|H_p u\|_{L^p} + \|u\|_{L^p} \right), \quad u \in D(H_p).
\]

**Proof.** The assertion follows from Proposition 4.2 and Corollary 4.3. \(\square\)

**Proposition 4.5.** (i) If \(2 < p < \infty\) and \(0 < |\text{Im} \lambda| < 1 - \frac{2}{p}\), then \(N(\lambda + H_p) \neq \{0\}\), and then

\[
\left\{ \lambda \in C ; |\text{Im} \lambda| \leq 1 - \frac{2}{p} \right\} \subset \sigma(H_p);
\]

(ii) If \(1 < p < 2\) and \(0 < |\text{Im} \lambda| < \frac{2}{p} - 1\), then \(N(\lambda + H_p) \subset L^p\), and then

\[
\left\{ \lambda \in C ; |\text{Im} \lambda| \leq \frac{2}{p} - 1 \right\} \subset \sigma(H_p).
\]
This contradicts the fact the closure of Volume 3, Issue 1, 21–34.

In other words, using the operators we can rewrite

\[ \square \]

we obtain the assertion.

(iii) \( (1 < p < 2, \Im \lambda < \frac{2}{p} - 1) \) Note that \( H_p \) is the adjoint operator of \( H_{p'} \). Since \( w_1 \in D(H_{p'}) \) for every \( u \in C_0^\infty(\mathbb{R}) \),

\[
\int_{-\infty}^{\infty} (\lambda u + H_p u) w_1 \, dx = \int_{-\infty}^{\infty} u (\lambda w_1 + H_{p'} w_1) \, dx = 0,
\]

the closure of \( R(\lambda + H_p) \) does not coincide with \( L^p \), that is, \( \overline{R(\lambda + H_p)} \subsetneq L^p \).

Since \( \sigma(H_p) \) is closed in \( \mathbb{C} \) and we can argue the same assertion for \( \Im \lambda < 0 \) via complex conjugation, we obtain the assertion.

Combining the assertions above, we finally obtain Theorem 1.1.

5. Absence of \( C_0 \)-semigroups on \( L^p \) \( (p \neq 2, V = 0) \)

In Theorem 1.1, we do not prove any assertions related to generation of \( C_0 \)-semigroups by \( \pm iH_p \).

In this subsection we prove

**Theorem 5.1.** Neither \( iH_p \) nor \( -iH_p \) generates \( C_0 \)-semigroup on \( L^p \).

**Proof.** We argue by a contradiction. Assume that \( iH_p \) generates a \( C_0 \)-semigroup \( T(t) \) on \( L^p \). Then it follows from Theorem 1.1 (the coincidence of resolvent operators) that we have \( T(t)f = S(t)f \) for every \( t > 0 \) and \( f \in L^2 \cap L^p \), where \( S(t) \) is the \( C_0 \)-group generated by the skew-adjoint operator \( iH_2 \).

Fix \( f_0 \in L^2 \cap L^p \) such that \( \mathcal{F} f_0 \notin L^p \) (\( \mathcal{F} \) is the Fourier transform). Then by the Mehler’s formula (see e.g., Cazenave [3, Remark 9.2.5]), we see that

\[
[S(t)]f(x) = \left( \frac{1}{2\pi \sinh(2t)} \right)^{\frac{n}{2}} e^{-\frac{\lambda^2}{\sinh(2t)}} \int_{-\infty}^{\infty} e^{-\frac{i}{\sinh(2t)} \cdot x} e^{-\frac{i\lambda}{\sinh(2t)} |y|^2} f(y) \, dy.
\]

In other words, using the operators

\[
M_x g(x) := e^{-\frac{|x|^2}{\pi}} g(x), \quad D_x g(x) := \tau^{-\frac{n}{2}} g(\tau^{-1} x),
\]

we can rewrite \( S(t) \) as the following form \( S(t)f = M_{\tanh(2t)} D_{\sinh(2t)} M_{\tanh(2t)} f \). Taking \( f_0 = M_{\tanh(2t)} D_{\sinh(2t)} f_0 \in L^p \), we have

\[
S(t_0) f_0 = M_{\tanh(2t_0)} D_{\sinh(2t_0)} f_0 \in L^p.
\]

This contradicts the fact \( T(t_0) f_0 \notin L^p \). This completes the proof.
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Conflict of Interest

All authors declare no conflicts of interest in this paper.

References


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