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# $H_\infty$ Disturbance Attenuation of Nonlinear Networked Control Systems via Takagi-Sugeno Fuzzy Model

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**Abstract:** $H_\infty$ disturbance attenuation of nonlinear networked systems which are described by the Takagi-Sugeno fuzzy time-delay systems is concerned. In the networked control system, the control signal is delayed and the closed-loop system with the controller can be modeled as a fuzzy system with time-varying delays in sensor and actuator nodes. The system often encounters the external noises that disturb its behaviors. For such a nonlinear system with delays, the $H_\infty$ disturbance attenuation problem is considered. Multiple Lyapunov-Krasovskii function with multiple integral functions allows us to obtain less conservative conditions for a networked control system to satisfy the disturbance attenuation criterion. Based on this approach, a novel control design method for a networked control system is proposed. An illustrative example is given to show the effectiveness of the proposed method.

**Keywords:** Takagi-Sugeno fuzzy systems, nonlinear systems, observer dessgin, linear matrix inequality

## 1. Introduction

Networked control systems(NCSs) are a class of systems where the signals of feedback loops are closed via communication network. These systems are found in many applications such as automobiles and airplanes, large scale disributed industrial systems and telecommunication systems due to easier installation and maintenance, simpler upgrading and more reliability over the point-to-point connected systems([3]). Therefore, much attention has been paid to NCSs in the last decades([5], [21]). In the networked control system, the information is exchanged with packets through a network where the data packets encounter delays. Considering the effects of network-induced delays in nonlinear NCS, we model its closed-loop system as a fuzzy system with bounded delays.

For a nonlinear control system, Takagi-Sugeno fuzzy model has been playing an important role. It can represent a nonlinear system effectively and is known to be a great tool to analyze and synthesize nonlinear control systems([11], [12], [13]). The papers [4], [6], [7], [9], [10], [16], [19], [20], and [22] considered control design problems for nonlinear networked control systems. The paper [6] partially introduced a multiple Lyapunov-Krasovskii matrix method for fuzzy systems with time-delay but it is not a general multiple Lyapunov matrix method. The papers [4], [9], [20], and [22] discussed various fuzzy networked control systems but all employed a common Lyapunov-Krasovskii function method. The papers [7], [10], and [19] employed a common Lyapunov-Krasovskii function method with descriptor system approach, which is still more conservative than a multiple Lyapunov-Krasovskii matrix method. The papers [6], [16], and [19] used a free matrix method to reduce the conservatism but increase computational load by introducing a number of free matrices. Furthermore, the paper [17] introduced a new multiple Lyapunov matrix method but only considered the stability of a networked control system. The papers [17] and [18] considered the stability and stabilization problems based on multiple Lyapunov-Krasovskii matrix method.

In this paper, we consider the $H_\infty$ disturbance attenuation of nonlinear networked control systems based on Takagi-Sugeno fuzzy models. First, we assume a new class of fuzzy feedback controller and consider the $H_\infty$ disturbance attenuation of the closed-loop system with such a feedback controller. In order to obtain less conservative $H_\infty$ disturbance attenuation conditions, we introduce a new type of multiple Lyapunov-Krasovskii function, which reduces the conservatism in stability conditions. A multiple Lyapunov-Krasovskii function is a natural extension of a common Lyapunov-Krasovskii function. However, a conventional multiple Lyapunov function contains the membership function and hence a resulting condition depends on the derivatives of the membership function. However, the derivative of the membership function may not always be known a priori nor differentiable. The paper [8] introduced a new class of multiple Lyapunov function, which contains an integral of the membership function of fuzzy systems. This approach requires no information on the derivatives of the membership function and is shown to reduce the conservatism in $H_\infty$ disturbance attenuation conditions. In addition, triple and quadruple integrals of Lyapunov-Krasovskii functions are employed, which enormously reduce the conservatism. Based on such a multiple Lyapunov-Krasovskii function, a control design method of nonlinear networked control systems are proposed. Finally, a numerical example is shown to illustrative our control design method and to show the effectiveness of our approach.

## 2. Fuzzy Model of Networked Control Systems

Consider the Takagi-Sugeno fuzzy model, described by the following IF-THEN rules:

$$\begin{array}{ll} IF & \xi_1 \ is \ M_{i1} \ and \ \cdots \ and \ \xi_p \ is \ M_{ip}, \\ THEN & \dot{x}(t) = A_i x(t) + B_i u(t) + D_i w(t), \\ & z(t) = C_i x(t) \end{array}$$

where $x(t) \in \Re^n$ is the state, $u(t) \in \Re^m$ is the control input. and $z(t) \in \Re^q$ is the controlled output. The matrices $A_i$, $B_i, C_i$ and $D_i$ are constant matrices of appropriate dimensions. $r$ is the number of IF-THEN rules. $M_{ij}$ are fuzzy sets and $\xi_1, \cdots, \xi_p$ are premise variables. We set $\xi = [\xi_1 \quad \cdots \quad \xi_p]^T$. The premise variable $\xi(t)$ is assumed to be measurable.

Then, the state equation and the controlled output equation are described by

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r} \lambda_i(\xi)\{A_i x(t) + B_i u(t) + D_i w(t)\} \\ &\triangleq A_\lambda x(t) + B_\lambda u(t) + D_\lambda w(t) \end{aligned} \tag{2.1}$$

$$\begin{aligned} z(t) &= \sum_{i=1}^{r} \lambda_i(\xi) C_i x(t) \\ &\triangleq C_\lambda x(t) \end{aligned} \tag{2.2}$$

where

$$\lambda_i(\xi) = \frac{\beta_i(\xi)}{\sum_{i=1}^{r} \beta_i(\xi)}, \quad \beta_i(\xi) = \prod_{j=1}^{p} M_{ij}(\xi_j)$$

and $M_{ij}(\cdot)$ is the grade of the membership function of $M_{ij}$. We assme

$$\lambda_i(\xi(t)) \geq 0, \ i = 1, \cdots, r, \ \sum_{i=1}^{r} \lambda_i(\xi(t)) = 1 \tag{2.3}$$

for any $\xi(t)$.

In the considered networked control system, the controller and the actuator are event-driven and sampler is clock-driven. The actual input of the system (2.1) is realized via a zero-order hold device. The sampling period is assumed to be a positive constant $T$ and the information of the zero-order hold may be updated between sampling instants. The updating instants of the zero-order hold are denoted by $t_k$, and $\tau_a$ and $\tau_b$ are the time-delays from the sampler to the controller and from the controller to the zero-order hold at the updating instant $t_k$, respectively. So, the successfully transmitted data in the networked control system at the instant $t_k$ experience round trip delay $\tau = \tau_a + \tau_b$ which does not need to be restricted inside one sampling period. Regarding the role of the zero-order hold, for a state sample data $t_k - \tau$, the corresponding control signal would act on the plant from $t_k$ unto $t_{k+1}$. Therefore, the rules of the fuzzy control input for $t_k \leq t \leq t_{k+1}$, is written as follows:

$$\begin{array}{ll} IF & \xi_1 \ is \ M_{i1} \ and \ \cdots \ and \ \xi_p \ is \ M_{ip}, \\ THEN & u(t) = K_i x(t - \tau(t)), \ \ i = 1, \cdots, r. \end{array}$$

where $K_i, \ i = 1, \cdots, r$ are constant matrices, and $\tau(t)$ may be an unknown time varying delay but its lower bound $\tau_1$ and upper bound $\tau_2$ are assumed to be known. The upper bound $\eta$ of the delay rate is also assumed to be known:

$$\tau_1 \leq \tau(t) \leq \tau_2, \ 0 < \dot{\tau}(t) \leq \eta.$$

Then, an overall controller is given by

$$u(t) = \sum_{i=1}^{r} \mu_i(\xi(t - \tau(t))) K_i x(t - \tau(t))$$

$$\overset{\Delta}{=} \quad K_\mu^\tau x(t-\tau(t)) \tag{2.4}$$

where

$$\mu_i(\xi(t)) = \frac{1}{h}\int_{t-h}^{t} \lambda_i(\xi(s))ds,$$

and $h > 0$ is some scalar. The closed-loop system (2.1) with (2.4) is given by

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{r}\sum_{l=1}^{r}\lambda_i(\xi(t))\mu_l(\xi(t-\tau(t)))\{A_i x(t) + B_i K_l x(t-\tau(t)) + D_i w(t)\} \\
&= A_\lambda x(t) + B_\lambda K_\mu^\tau x(t-\tau(t) + D_\lambda w(t). \qquad (2.5)
\end{aligned}$$

We note that $\mu_i(\xi(t)) \geq 0,\ i = 1, \cdots, r$ and

$$\begin{aligned}
\sum_{i=1}^{r}\mu_i(\xi(t)) &= \frac{1}{h}\int_{t-h}^{t}\sum_{i=1}^{r}\lambda_i(\xi(s))ds \\
&= 1,
\end{aligned}$$

which imply that $\mu_i(\xi(t))$ and $\lambda_i(\xi(t))$ share the same properties as seen in (2.3).

We define the cost function

$$J = \int_0^\infty (z^T(t)z(t) - \gamma^2 w^T(t)w(t))dt \tag{2.6}$$

where $\gamma$ is a prescribed scalar. Our problem is to find a condition such that the closed-loop system (2.2) and (2.5) is asymptotically stable with $w(t) = 0$ and it satisfies $J < 0$ in (2.6). In this case, the system is said to achieve the $\mathrm{H}_\infty$ disturbance attenuation with $\gamma$.

## 3. $\mathrm{H}_\infty$ Disturbance Attenuation

Let us first assume that all the controller gain matrices $K_i,\ i = 1, \cdots, r$ are given. Importance on the disturbance attenuation conditions lies on how to choose an appropriate Lyapunov-Krasovskii function. Here, we introduce a new Lyapunov-Krasovskii function. To begin with, let us consider a polytopic matrix:

$$Z_\mu = \sum_{i=1}^{r}\mu_i(\xi(t))Z_i$$

and similar notations will be used for other matrices. It is easy to see that the time-derivative of $Z_\mu$ is calculated as

$$\begin{aligned}
\dot{Z}_\mu &= \sum_{i=1}^{r}\dot{\mu}_i(\xi(t))Z_i \\
&= \frac{1}{h}\sum_{i=1}^{r}(\lambda_i(\xi(t)) - \lambda_i(\xi(t-\tau)))Z_i \\
&\overset{\Delta}{=} \frac{1}{h}(\mathrm{Z}_\lambda - Z_\lambda^\tau). \qquad (3.1)
\end{aligned}$$

For later use, we give some notation and lemmas:

$$\begin{aligned}\zeta(t) &= \Big[x^T(t) \quad x^T(t-\tau(t)) \quad x^T(t-\tau_1) \quad x^T(t-\tau_2) \quad \int_{t-\tau_1}^{t} x^T(s)ds \quad \int_{t-\tau(t)}^{t-\tau_1} x^T(s)ds \\ &\quad \int_{t-\tau_2}^{t-\tau(t)} x^T(s)ds \quad \int_{-\tau_1}^{0}\int_{t+\beta}^{t} x^T(s)dsd\beta \quad \int_{-\tau(t)}^{-\tau_1}\int_{t+\beta}^{t} x^T(s)ds \quad \int_{-\tau_2}^{-\tau(t)}\int_{t+\beta}^{t} x^T(s)ds \quad w(t)\Big]^T \\ &\triangleq \Big[\zeta_1^T(t) \quad \zeta_2^T(t) \quad \cdots \quad \zeta_{11}^T(t)\Big]^T.\end{aligned}$$

**Lemma 3.1.** *(Jensen's Inequality) For $\tau \in \Re$, $x(t) \in \Re^n$, and $P > 0 \in \Re^{n\times n}$, the following inequalities hold:*

$$\begin{aligned}&-\tau\int_{t-\tau}^{t} x^T(s)Px(s)ds \le \int_{t-\tau}^{t} x^T(s)dsP\int_{t-\tau}^{t} x(s)ds,\\ &-\frac{\tau^2}{2}\int_{-\tau}^{0}\int_{t+\beta}^{t} x^T(s)Px(s)dsd\beta \le \int_{-\tau}^{0}\int_{t+\beta}^{t} x^T(s)dsd\beta P\int_{-\tau}^{0}\int_{t+\beta}^{t} x(s)dsd\beta,\\ &-\frac{\tau^3}{6}\int_{-\tau}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t} x^T(s)Px(s)dsd\beta d\theta \le \int_{-\tau}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t} x^T(s)dsd\beta d\theta P\int_{-\tau}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t} x(s)dsd\beta d\theta.\end{aligned}$$

**Lemma 3.2.** *([1]) For $\tau_1,\ \tau_2,\ \alpha,\ \varepsilon \in \Re$, $x(t) \in \Re^n$, and $P > 0 \in \Re^{n\times n}$, the following inequalities hold:*

$$\begin{aligned}&-(\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau_1} x^T(s)Px(s)ds \le -\zeta_6^T(t)P\zeta_6(t) - \zeta_7^T(t)P\zeta_7(t) - (1-\alpha)\zeta_6^T(t)P\zeta_6(t) - \alpha\zeta_7^T(t)P\zeta_7(t),\\ &-\frac{(\tau_2^2-\tau_1^2)}{2}\int_{t-\tau_2}^{t-\tau_1}\int_{t+\beta}^{t} x^T(s)Px(s)dsd\beta \le -\zeta_9^T(t)P\zeta_9(t) - \zeta_{10}^T(t)P\zeta_{10}(t) - (1-\varepsilon)\zeta_9^T(t)P\zeta_9(t) - \varepsilon\zeta_{10}^T(t)P\zeta_{10}(t).\end{aligned}$$

Now, we are ready to give our first result.

**Theorem 3.1.** *Given control gain matrices $K_l,\ l = 1,\cdots,r$ and scalar $h > 0$. The closed-loop system (2.5) achieves the $H_\infty$ disturbance attenuation with $\gamma$ if there exist matrices $Z_j > 0,\ P_1 > 0,\ P_2 > 0,\ P_3 > 0,\ P_4 > 0,\ R_{j1} > 0,\ R_2 > 0,\ R_{3j} > 0,\ R_4 > 0,\ X_{1j} > 0,\ X_2 > 0,\ X_{3j} > 0,\ X_4 > 0,\ U_1 > 0,\ U_2 > 0,\ W_j,\ j = 1,\cdots,r$, and scalars $\delta_i > 0,\ i = 1,2$ such that*

$$\begin{bmatrix}\frac{1}{2}\theta_{ijl}+\theta_{1j}+\delta_1 I & C_i^T\\ * & -I\end{bmatrix} < 0,\ i,j,l = 1,\cdots,r, \tag{3.2}$$

$$\begin{bmatrix}\frac{1}{2}\theta_{ijl}+\theta_{2j}+\delta_1 I & C_i^T\\ * & -I\end{bmatrix} < 0,\ i,j,l = 1,\cdots,r, \tag{3.3}$$

$$\begin{bmatrix}\frac{1}{2}\theta_{ijl}+\theta_{3j}-\delta_2 I & C_i^T\\ * & -I\end{bmatrix} < 0,\ i,j,l = 1,\cdots,r, \tag{3.4}$$

$$\begin{bmatrix}\frac{1}{2}\theta_{ijl}+\theta_{4j}-\delta_2 I & C_i^T\\ * & -I\end{bmatrix} < 0,\ i,j,l = 1,\cdots,r, \tag{3.5}$$

$$\delta_1-\delta_2 > 0 \tag{3.6}$$

$$\begin{bmatrix}\frac{1}{\tau_1}Z_i+X_2 & -X_2\\ -X_2 & Q_{1i}+X_2\end{bmatrix} \ge 0,\ i = 1,\cdots,r, \tag{3.7}$$

$$\begin{bmatrix}\frac{1}{\tau_2-\tau_1}Z_i+X_4 & -X_4\\ -X_4 & Q_{2i}+X_4\end{bmatrix} \ge 0,\ i = 1,\cdots,r \tag{3.8}$$

*where* $\tau_{12} = \tau_2 - \tau_1,\ \tau_{12}^{(2)} = \tau_2^2 - \tau_1^2$

$$
\begin{aligned}
\theta_{1j} = & -e_7^T X_{3j} e_7 - (e_2 - e_4)^T X_4 (e_2 - e_4),\\
\theta_{2j} = & -e_6^T X_{3j} e_6 - (e_2 - e_3)^T X_4 (e_2 - e_3),\\
\theta_{3j} = & -e_{10}^T R_{3j} e_{10} - (\tau_{12} e_1 - e_7)^T R_4 (\tau_{12} e_1 - e_7),\\
\theta_{4j} = & -e_9^T R_{3j} e_9 - (\tau_{12} e_1 - e_6)^T R_4 (\tau_{12} e_1 - e_6),\\
\theta_{ijl} = & \pi_{ijl} - e_5^T X_1 e_5 - (e_1 - e_3)^T X_2 (e_1 - e_3) - e_6^T X_3 e_6 - e_7^T X_3 e_7 - (e_2 - e_3)^T X_4 (e_2 - e_3)\\
& -(e_2 - e_4)^T X_4 (e_2 - e_4) - e_8^T R_{1j} e_8 - (\tau_1 e_1 - e_5)^T R_2 (\tau_1 e_1 - e_5) - e_9^T R_{3j} e_9\\
& -e_{10}^T R_{3j} e_{10} - (\tau_{12} e_1 - e_6)^T R_4 (\tau_{12} e_1 - e_6) - (\tau_{12} e_1 - e_7)^T R_4 (\tau_{12} e_1 - e_7)\\
& -(\tfrac{\tau_{12}}{2} e_1 - e_8)^T U_1 (\tfrac{\tau_{12}}{2} e_1 - e_8) - (\tfrac{\tau_{12}^{(2)}}{2} e_1 - e_9 - e_{10})^T U_2 (\tfrac{\tau_{12}^{(2)}}{2} e_1 - e_9 - e_{10})
\end{aligned}
$$

$$
\pi_{ijl} = \begin{bmatrix}
\Lambda_{11ij} & \Lambda_{12ijl} & 0 & 0 & P_1 & 0 & 0 & \tau_1 P_3 & \tau_{12} P_4 & \tau_{12} P_4 & Z_j D_i + A_i^T \Omega D_i\\
* & \Lambda_{22ijl} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_l^T B_i^T \Omega D_i\\
* & * & -Q_{1j} + Q_{2j} & 0 & -P_1 & P_2 & P_2 & 0 & 0 & 0 & 0\\
* & * & * & -Q_{2j} & 0 & -P_2 & -P_2 & 0 & 0 & 0 & 0\\
* & * & * & * & 0 & 0 & 0 & -P_3 & 0 & 0 & 0\\
* & * & * & * & * & 0 & 0 & 0 & -P_4 & -P_4 & 0\\
* & * & * & * & * & * & 0 & 0 & -P_4 & -P_4 & 0\\
* & * & * & * & * & * & * & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & D_i^T \Omega D_i - \gamma^2 I
\end{bmatrix},
$$

$$
\begin{aligned}
\Lambda_{11ij} &= A_i^T Z_j + Z_j A_i + Q_{1j} + W_j + \tfrac{1}{h}(Z_i - Z_l) + \tau_1^2 X_{1j} + \tau_{12}^2 X_{3j} + \tfrac{\tau_1^4}{4} R_{1j} + \tfrac{(\tau_{12}^{(2)})^2}{4} R_{3j} + A_i^T \Omega A_i,\\
\Lambda_{12ijl} &= Z_j B_i K_l + A_i^T \Omega B_i K_l,\\
\Lambda_{22ijl} &= -(1 - \eta) W_j + K_l^T B_i^T \Omega B_i K_l,\\
\Omega &= \tau_1^2 X_2 + \tau_{12}^2 X_4 + \tfrac{\tau_1^4}{4} R_2 + \tfrac{(\tau_{12}^{(2)})^2}{4} R_4 + \tfrac{\tau_1^6}{36} U_1 + \tfrac{(\tau_2^3 - \tau_1^3)^2}{36} U_2,\\
\Phi_{il} &= \begin{bmatrix} A_i^T & K_l^T B_i^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_i^T \end{bmatrix},\\
\bar{C}_i &= \begin{bmatrix} C_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned}
$$

*and* $e_i,\ i = 1, \cdots, 11$ *denote an* 11*-dimensional fundamental vector whose i-th element is* 1 *and* 0 *elsewhere.*

**Proof:** Consider the following Lyapunov-Krasovskii function:

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t) + V_4(x_t) + V_5(x_t) \tag{3.9}$$

where $x_t = x(t+\theta),\ -\tau_2 \le \theta \le 0$,

$$\begin{aligned}
V_1(x_t) = {} & x^T(t)Z_\mu x(t) + \int_{t-\tau_1}^{t} x^T(s)dsP_1 \int_{t-\tau_1}^{t} x^T(s)ds + \int_{t-\tau_2}^{t-\tau_1} x^T(s)dsP_2 \int_{t-\tau_2}^{t-\tau_1} x(s)ds \\
& + \int_{-\tau_1}^{0}\int_{t+\theta}^{t} x^T(s)dsd\theta P_3 \int_{-\tau_1}^{0}\int_{t+\theta}^{t} x(s)dsd\theta + \int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} x^T(s)dsd\theta P_4 \int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} x(s)dsd\theta, \\
V_2(x_t) = {} & \int_{t-\tau_1}^{t} x^T(s)Q_{1\mu}x(s)ds + \int_{t-\tau_2}^{t-\tau_1} x^T(s)Q_{2\mu}x(s)ds + \int_{t-\tau(t)}^{t} x^T(s)W_\mu x(s)ds, \\
V_3(x_t) = {} & \tau_1 \int_{-\tau_1}^{0}\int_{t+\theta}^{t} x^T(s)X_{1\mu}x(s)dsd\theta + \tau_1 \int_{-\tau_1}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)X_2\dot{x}(s)dsd\theta \\
& +(\tau_2-\tau_1)\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} x^T(s)X_{3\mu}x(s)dsd\theta + (\tau_2-\tau_1)\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} \dot{x}^T(s)X_4\dot{x}(s)dsd\theta,
\end{aligned}$$

$$\begin{aligned}
V_4(x_t) = {} & \frac{\tau_1^2}{2}\int_{-\tau_1}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t} x^T(s)R_{1\mu}x(s)dsd\theta d\beta + \frac{\tau_1^2}{2}\int_{-\tau_1}^{0}\int_{\beta}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R_2\dot{x}(s)dsd\theta d\beta \\
& +\frac{\tau_2^2-\tau_1^2}{2}\int_{-\tau_2}^{-\tau_1}\int_{\beta}^{0}\int_{t+\theta}^{t} x^T(s)R_{3\mu}x(s)dsd\theta d\beta + \frac{\tau_2^2-\tau_1^2}{2}\int_{-\tau_2}^{-\tau_1}\int_{\beta}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R_4\dot{x}(s)dsd\theta d\beta, \\
V_5(x_t) = {} & \frac{\tau_1^3}{6}\int_{-\tau_1}^{0}\int_{\beta}^{0}\int_{\lambda}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)U_1\dot{x}(s)dsd\lambda d\beta d\theta \\
& +\frac{\tau_2^3-\tau_1^3}{6}\int_{-\tau_2}^{-\tau_1}\int_{\beta}^{0}\int_{\lambda}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)U_2\dot{x}(s)dsd\lambda d\beta d\theta
\end{aligned}$$

where

$$X_{j\mu} = \sum_{i=1}^{r}\mu_i(\xi)X_{ji} > 0,\ j = 1,\cdots,r$$

and similar notations are used. Now, we take the derivative of $V(x_t)$ with respect to $t$ along the solution of the system (2.5).

First, using Lemma 3.1, we see that

$$\begin{aligned}
\int_{t+\theta}^{t} \dot{x}^T(s)X_2\dot{x}(s)ds \ \ge\ & -\frac{1}{\theta}\int_{t+\theta}^{t} \dot{x}^T(s)dsX_2\int_{t+\theta}^{t} \dot{x}(s)ds \\
=\ & -\frac{1}{\theta}[x(t)-x(t+\theta)]^T X_2[x(t)-x(t+\theta)]
\end{aligned}$$

and

$$\begin{aligned}
\int_{-\tau_1}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)X_2\dot{x}(s)dsd\theta \ \ge\ & -\int_{-\tau_1}^{0}\frac{1}{\theta}[x(t)-x(t+\theta)]^T X_2[x(t)-x(t+\theta)]d\theta \\
=\ & \int_{0}^{\tau_1}\frac{1}{\theta}[x(t)-x(t-s)]^T X_2[x(t)-x(t-s)]ds \\
\ge\ & \frac{1}{\tau_1}\int_{0}^{\tau_1}[x(t)-x(t-s)]^T X_2[x(t)-x(t-s)]ds \\
=\ & \frac{1}{\tau_1}\int_{t-\tau_1}^{t}[x(t)-x(\alpha)]^T X_2[x(t)-x(\alpha)]d\alpha
\end{aligned}$$

Similarly, we have

$$\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t}\dot{x}^T(s)X_4\dot{x}(s)dsd\theta \;\geq\; \frac{1}{\tau_2-\tau_1}\int_{t-\tau_2}^{t-\tau_1}[x(t)-x(\alpha)]^TX_4[x(t)-x(\alpha)]d\alpha$$

Hence, we get

$$\begin{aligned}
& x^T(t)Z_\mu x(t)+\int_{t-\tau_1}^{t}x^T(s)Q_{1\mu}x(s)ds+\int_{t-\tau_2}^{t-\tau_1}x^T(s)Q_{2\mu}x(s)ds\\
& +\tau_1\int_{-\tau_1}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)X_2\dot{x}(s)dsd\theta+(\tau_2-\tau_1)\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t}\dot{x}^T(s)X_4\dot{x}(s)dsd\theta\\
\geq\; & \int_{t-\tau_1}^{t}\begin{bmatrix} x(t)\\ x(\alpha)\end{bmatrix}^T\begin{bmatrix}\frac{1}{\tau_1}Z_\mu+X_2 & -X_2\\ -X_2 & Q_{1\mu}+X_2\end{bmatrix}\begin{bmatrix} x(t)\\ x(\alpha)\end{bmatrix}d\alpha\\
& +\int_{t-\tau_2}^{t-\tau_1}\begin{bmatrix} x(t)\\ x(\alpha)\end{bmatrix}^T\begin{bmatrix}\frac{1}{\tau_2-\tau_1}Z_\mu+X_4 & -X_4\\ -X_4 & Q_{2\mu}+X_4\end{bmatrix}\begin{bmatrix} x(t)\\ x(\alpha)\end{bmatrix}d\alpha
\end{aligned}$$

It follows from the above that for $V_1(x_t)+V_2(x_t)+V_3(x_t)$ to be positive, the positive definiteness of $Q_{1i}$ and $Q_{2i},\ i=1,\cdots,r$ can be removed if the positive definiteness of $P_i, W_j, X_{1j}, X_{3j},\ i=1,\cdots,4,\ j=1,\cdots,r$ is guaranteed and (3.7)-(3.8) are satisfied.

The derivatives of $V_1(x_t)$ and $V_2(x_t)$ in (3.9) are calculated as follows:

$$\begin{aligned}
\dot{V}_1(x_t) \;=\; & 2(A_\lambda x(t)+B_\lambda K_\mu^\tau x(t-\tau(t))+D_\lambda w(t))^TZ_\mu x(t)+\frac{1}{h}x^T(t)(Z_\lambda-Z_\lambda^\tau)x(t)\\
& +2(x(t)-x(t-\tau_1))^TP_1\int_{t-\tau_1}^{t}x(s)ds+2(x(t-\tau_1)-x(t-\tau_2))^TP_2\int_{t-\tau_2}^{t-\tau_1}x(s)ds\\
& +2[\tau_1x(t)-\int_{t-\tau_1}^{t}x^T(s)ds]^TP_3\int_{-\tau_1}^{0}\int_{t+\theta}^{t}x(s)dsd\theta\\
& +2[(\tau_2-\tau_1)x(t)-\int_{t-\tau_2}^{t-\tau_1}x^T(s)ds]^TP_4\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t}x(s)dsd\theta, \qquad (3.10)\\
\dot{V}_2(x_t) \;\leq\; & x^T(t)(Q_{1\mu}+W_\mu)x(t)-x^T(t-\tau_1)Q_{1\mu}x(t-\tau_1)+x^T(t-\tau_1)Q_{2\mu}x(t-\tau_1)\\
& -x^T(t-\tau_2)Q_{2\mu}x(t-\tau_2)-(1-\eta)x^T(t-\tau(t))W_\mu x(t-\tau(t)). \qquad (3.11)
\end{aligned}$$

Using Lemmas 3.1 and 3.2, we have

$$\begin{aligned}
\dot{V}_3(x_t) \;=\; & \tau_1^2x^T(t)X_{1\mu}x(t)-\tau_1\int_{t-\tau_1}^{t}x^T(s)X_{1\mu}x(s)ds+\tau_1^2\dot{x}^T(t)X_2\dot{x}(t)-\tau_1\int_{t-\tau_1}^{t}\dot{x}^T(s)X_2\dot{x}(s)ds\\
& +(\tau_2-\tau_1)^2x^T(t)X_{3\mu}x(t)-(\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau_1}x^T(s)X_{3\mu}x(s)ds\\
& +(\tau_2-\tau_1)^2\dot{x}^T(t)X_4\dot{x}(t)-(\tau_2-\tau_1)\int_{t-\tau_2}^{t-\tau_1}\dot{x}^T(s)X_4\dot{x}(s)ds,\\
\leq\; & \tau_1^2x^T(t)X_{1\mu}x(t)-\zeta_5^T(t)X_{1\mu}\zeta_5(t)+\tau_1^2\dot{x}^T(t)X_2\dot{x}(t)-(\zeta_1(t)-\zeta_3(t))^TX_2(\zeta_1(t)-\zeta_3(t))\\
& +(\tau_2-\tau_1)^2x^T(t)X_{3\mu}x(t)-\zeta_6^T(t)X_{3\mu}\zeta_6(t)-\zeta_7^T(t)X_{3\mu}\zeta_7(t)-(1-\alpha)\zeta_6^T(t)X_{3\mu}\zeta_6(t)\\
& -\alpha\zeta_7^T(t)X_{3\mu}\zeta_7(t)+(\tau_2-\tau_1)^2\dot{x}^T(t)X_4\dot{x}(t)-(\zeta_2(t)-\zeta_3(t))^TX_4(\zeta_2(t)-\zeta_3(t))
\end{aligned}$$

$$
\begin{aligned}
& -(\zeta_2(t)-\zeta_4(t))^T X_4(\zeta_2(t)-\zeta_4(t)) - (1-\alpha)(\zeta_2(t)-\zeta_3(t))^T X_4(\zeta_2(t)-\zeta_3(t)) \\
& -\alpha(\zeta_2(t)-\zeta_4(t))^T X_4(\zeta_2(t)-\zeta_4(t)), \qquad (3.12) \\
\dot{V}_4(x_t) \;=\;& \frac{\tau_1^4}{4}x^T(t)R_{1\mu}x(t) - \frac{\tau_1^2}{2}\int_{-\tau_1}^{0}\int_{t+\beta}^{t} x^T(s)R_{1\mu}x(s)dsd\beta + \frac{\tau_1^4}{4}\dot{x}^T(t)R_2\dot{x}(t) \\
& -\frac{\tau_1^2}{2}\int_{-\tau_1}^{0}\int_{t+\beta}^{t}\dot{x}^T(s)R_2\dot{x}(s)dsd\beta + \frac{(\tau_2^2-\tau_1^2)^2}{4}x^T(t)R_{3\mu}x(t) - \frac{\tau_2^2-\tau_1^2}{2}\int_{-\tau_2}^{-\tau_1}\int_{t+\beta}^{t}x^T(s)R_{3\mu}x(s)dsd\beta \\
& +\frac{(\tau_2^2-\tau_1^2)^2}{4}\dot{x}^T(t)R_4\dot{x}(t) - \frac{\tau_2^2-\tau_1^2}{2}\int_{-\tau_2}^{-\tau_1}\int_{t+\beta}^{t}\dot{x}^T(s)R_4\dot{x}(s)dsd\beta \\
\leq\;& \frac{\tau_1^4}{4}x^T(t)R_{1\mu}x(t) - \zeta_8^T(t)R_{1\mu}\zeta_8(t) + \frac{\tau_1^4}{4}\dot{x}^T(t)R_2\dot{x}(t) - (\tau_1\zeta_1(t)-\zeta_5(t))^T(t)R_2(\tau_1\zeta_1(t)-\zeta_5(t)) \\
& +\frac{(\tau_2^2-\tau_1^2)^2}{4}x^T(t)R_{3\mu}x(t) - \zeta_9^T(t)R_{3\mu}\zeta_9(t) - \zeta_{10}^T(t)R_{3\mu}\zeta_{10}(t) - (1-\varepsilon)\zeta_9^T(t)R_{3\mu}\zeta_9(t) \\
& -\varepsilon\zeta_{10}^T(t)R_{3\mu}\zeta_{10}(t) + \frac{(\tau_2^2-\tau_1^2)^2}{4}\dot{x}^T(t)R_4\dot{x}(t) \\
& -((\tau_2-\tau_1)\zeta_1(t)-\zeta_7(t))^T R_4((\tau_2-\tau_1)\zeta_1(t)-\zeta_7(t)) \\
& -((\tau_2-\tau_1)\zeta_1(t)-\zeta_6(t))^T R_4((\tau_2-\tau_1)\zeta_1(t)-\zeta_6(t)) \\
& -\varepsilon((\tau_2-\tau_1)\zeta_1(t)-\zeta_7(t))^T R_4((\tau_2-\tau_1)\zeta_1(t)-\zeta_7(t)) \\
& -(1-\varepsilon)((\tau_2-\tau_1)\zeta_1(t)-\zeta_6(t))^T R_4((\tau_2-\tau_1)\zeta_1(t)-\zeta_6(t)) \qquad (3.13) \\
\dot{V}_5(x_t) \;=\;& \frac{\tau_1^6}{36}\dot{x}^T(t)U_1\dot{x}(t) - \frac{\tau_1^3}{6}\int_{-\tau_1}^{0}\int_{\beta}^{0}\int_{t+\lambda}^{t}\dot{x}^T(s)U_1\dot{x}(s)dsd\lambda d\beta + \frac{(\tau_2^3-\tau_1^3)^2}{36}\dot{x}^T(t)U_2\dot{x}(t) \\
& -\frac{\tau_2^3-\tau_1^3}{6}\int_{-\tau_2}^{-\tau_1}\int_{\beta}^{0}\int_{t+\lambda}^{t}\dot{x}^T(s)U_2\dot{x}(s)dsd\lambda d\beta \\
\leq\;& \frac{\tau_1^6}{36}\dot{x}^T(t)U_1\dot{x}(t) - (\frac{\tau_1^2}{2}\zeta_1(t)-\zeta_8(t))^T U_1(\frac{\tau_1^2}{2}\zeta_1(t)-\zeta_8(t)) + \frac{(\tau_2^3-\tau_1^3)^2}{36}\dot{x}^T(t)U_2\dot{x}(t) \\
& -(\frac{\tau_2^2-\tau_1^2}{2}\zeta_1(t)-\zeta_9(t)-\zeta_{10}(t))^T U_2(\frac{\tau_2^2-\tau_1^2}{2}\zeta_1(t)-\zeta_9(t)-\zeta_{10}(t)). \qquad (3.14)
\end{aligned}
$$

It follows from (3.10)-(3.14) that

$$
\begin{aligned}
& \dot{V}(x_t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \\
=\;& \zeta^T(t)\left[\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r}\lambda_i(\xi)\mu_j(\xi)\mu_l(\xi(t-\tau))(\alpha\theta_{ijl}^{(1)} + (1-\alpha)\theta_{ijl}^{(2)} + \varepsilon\theta_{ijl}^{(3)} + (1-\varepsilon)\theta_{ijl}^{(4)}\right]\zeta(t) \\
& +x^T(t)(\sum_{i=1}^{r}\lambda_i(\xi)C_i)^T(\sum_{i=1}^{r}\lambda_i(\xi)C_i)x(t) + \dot{x}^T(t)\Omega\dot{x}(t) \\
\triangleq\;& \zeta^T(t)\left[(\alpha\theta_{\lambda\mu\mu}^{(1)} + (1-\alpha)\theta_{\lambda\mu\mu}^{(2)} + \varepsilon\theta_{\lambda\mu\mu}^{(3)} + (1-\varepsilon)\theta_{\lambda\mu\mu}^{(4)}\right]\zeta(t) + \zeta^T(t)e_1^T C_\lambda^T C_\lambda e_1\zeta(t) \\
& +\zeta^T(t)(A_\lambda e_1 + B_\lambda K_\mu^\tau e_2 + D_\lambda e_{11})^T\Omega(A_\lambda e_1 + B_\lambda K_\mu^\tau e_2 + D_\lambda e_{11})\zeta(t) \qquad (3.15)
\end{aligned}
$$

where $\theta_{ijl}^{(k)} = \frac{1}{2}\theta_{ijl} + \theta_{kj},\ k = 1,2$ and $\theta_{ijl}^{(k)} = \frac{1}{2}\theta_{ijl} + \theta_{kj},\ k = 3,4.$ By Schur complement formula, the upper bound of $\dot{V}$ is negative if and only if

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r}\lambda_i(\xi)\mu_j(\xi)\mu_l(\xi(t-\tau))\begin{bmatrix}\alpha\theta_{ijl}^{(1)}+(1-\alpha)\theta_{ijl}^{(2)}+\varepsilon\theta_{ijl}^{(3)}+(1-\varepsilon)\theta_{ijl}^{(4)} & \Phi_{il}^T & \bar{C}_i^T\\ * & -\Omega^{-1} & 0\\ * & * & -I\end{bmatrix}<0. \quad (3.16)$$

(3.16) holds if and only if the following conditions hold simultaneously provided that $\delta_2 < \delta_1$;

$$\alpha\Psi_{\lambda\mu\mu}^{(1)}+(1-\alpha)\Psi_{\lambda\mu\mu}^{(2)} < -\delta_1 I,$$
$$\varepsilon\Psi_{\lambda\mu\mu}^{(3)}+(1-\varepsilon)\Psi_{\lambda\mu\mu}^{(4)} < \delta_2 I$$

where

$$\Psi_{\lambda\mu\mu}^{(i)}=\begin{bmatrix}\theta_{\lambda\mu\mu}^{(i)} & \Phi_{\lambda\mu} & C_\lambda^T\\ * & -\Omega^{-1} & 0\\ * & * & -I\end{bmatrix},\ i=1,2,3,4.$$

The above conditions can be rewritten as

$$\alpha(\Psi_{\lambda\mu\mu}^{(1)}+\delta_1 I)+(1-\alpha)(\Psi_{\lambda\mu\mu}^{(2)}+\delta_1 I) \ < \ 0, \quad (3.17)$$
$$\varepsilon(\Psi_{\lambda\mu\mu}^{(3)}-\delta_2 I)+(1-\varepsilon)(\Psi_{\lambda\mu\mu}^{(4)}-\delta_2 I) \ < \ 0. \quad (3.18)$$

Since $0 \le \alpha,\ \varepsilon \le 1$, the terms $\alpha(\Psi_{\lambda\mu\mu}^{(1)}+\delta_1 I)+(1-\alpha)(\Psi_{\lambda\mu\mu}^{(2)}+\delta_1 I)$ is a convex combination of $\Psi_{\lambda\mu\mu}^{(1)}+\delta_1 I$ and $\Psi_{\lambda\mu\mu}^{(2)}+\delta_1 I$. Similarly, the terms $\varepsilon(\Psi_{\lambda\mu\mu}^{(3)}-\delta_2 I)+(1-\varepsilon)(\Psi_{\lambda\mu\mu}^{(4)}-\delta_2 I)$ is a convex combination of $\Psi_{\lambda\mu\mu}^{(3)}-\delta_2 I$ and $\Psi_{\lambda\mu\mu}^{(4)}-\delta_2 I$. These combinations are negative definite if the vertices become negative. Therefore, (3.17) and (3.18) are equivalent to

$$\Psi_{\lambda\mu\mu}^{(1)}+\delta_1 I \ < \ 0,$$
$$\Psi_{\lambda\mu\mu}^{(2)}+\delta_1 I \ < \ 0,$$
$$\Psi_{\lambda\mu\mu}^{(3)}-\delta_2 I \ < \ 0,$$
$$\Psi_{\lambda\mu\mu}^{(4)}-\delta_2 I \ < \ 0$$

which can be written as (3.2)-(3.5). It follows from (3.15) that this proves that the conditions (3.2)-(3.6) suffice to show

$$\dot{V}(x_t)+z^T(t)z(t)-\gamma^2 w^T(t)w(t)<0.$$

Integrating $t=0$to $t=\infty$, we have

$$V(x(\infty))-V(x(0))+J<0.$$

Since $V(x(\infty))\ge 0$ and $V(x(0))=0$, we can show that $J<0$ and this achieves the $H_\infty$ disturbance attenuation of the system (2.5). The stability of the system with $w(t)=0$ is proved in the same lines as in [18].

**Remark 3.1.** *The paper [18] uses the similar method to propose a stabilizing control design for nonlinear NCSs. It has shown that its method has advantaes over the previous methods in [6] and [7]. The novelty of Theorem 3.1 lies in a new multiple Lyapunov-Krasovskii function (3.9) where*

*$Z_j$, $W_j$, $Q_{1j}$, $Q_{2j}$, $R_{1j}$, $R_{3j}$, $X_{1j}$ and $X_{3j}$ are multiple Lyapunov matrices. In addition, the integral $\mu_i(\xi(t))$, $i = 1, \cdots, r$ of the membership functions avoid the derivatives of the membership function in the $H_\infty$ disturbance attenuation conditions (3.2)-(3.5). The quadruple integral terms and the quadratic forms of the double integral terms $\int\int x^T(s)dsd\beta P \int\int x^T(s)dsd\beta$ are employed in (3.9), which leads to a drastic reduction of the conservatism in the $H_\infty$ disturbance attenuation condition. In fact, recent papers [6] and [7] do not use the quadruple integrals and the quadratic forms of the double integrals. This implies that our $H_\infty$ disturbance attenuation conditions are less conservative than recent results, and is technically better than others. In fact, this advantage was shown in [18].*

**Remark 3.2.** *The conditions (3.7)-(3.8) remove the positive definiteness of $Q_{1i}$ and $Q_{2i}$, $i = 1, \cdots, r$, and reduce the conservatism in conditions in Theorem 3.1.*

**Remark 3.3.** *The conditions (3.2)-(3.8) are not strict LMIs unless $h > 0$ is given. By defining $\tilde{h} = \frac{1}{h}$, these conditions can be seen as bilinear matrix inequalities. Effective algorithms to solve them include the branch-and-cut algorithm, the branch-and-bound algorithm, and the Lagrangian dual global optimization algorithm in [2], [14], and [15], respectively.*

## 4. Control Design

Next, we shall propose a control design method. It is assumed that instead of the controller (2.4), a form of the controller is given by non-PDC, described by

$$\begin{aligned} u(t) &= \sum_{i=1}^{r} \mu_i(\xi(t-\tau(t)))K_i \left( \sum_{i=1}^{r} \mu_i(\xi(t-\tau(t)))Z_i \right)^{-1} x(t-\tau(t)) \\ &= K_\mu^\tau (Z_\mu^\tau)^{-1} x(t-\tau(t)) \end{aligned} \tag{4.1}$$

where $K_i$ and $Z_i$, $i = 1, \cdots, r$ are to be determined, and $\mu_i$, $i = 1, \cdots, r$ are given as in (2.4). Then, the closed-loop system (2.1) with (4.1) becomes

$$\dot{x}(t) = A_\lambda x(t) + B_\lambda K_\mu^\tau (Z_\mu^\tau)^{-1} x(t-\tau(t)) + D_\lambda w(t). \tag{4.2}$$

Applying Theorem 3.1, we obtain the following theorem for control design.

**Theorem 4.1.** *For some scalar $h > 0$. A controller (4.1) makes the fuzzy system (2.1)-(2.2) achieve the $H_\infty$ disturbance attenuation with $\gamma$ if there exist matrices $Z_j > 0$, $\bar{P}_{1mn} > 0$, $\bar{P}_{2mn} > 0$, $\bar{P}_{3mn} > 0$, $\bar{P}_{4mn} > 0$, $\bar{R}_{1jmn} > 0$, $\bar{R}_{2mn} > 0$, $\bar{R}_{3jmn} > 0$, $\bar{R}_{4mn} > 0$, $\bar{X}_{1jmn} > 0$, $\bar{X}_{2mn} > 0$, $\bar{X}_{3jmn} > 0$, $\bar{X}_{4mn} > 0$, $\bar{U}_{1mn} > 0$, $\bar{U}_{2mn} > 0$, $\bar{W}_{jmn}$, $K_j$, $j, m, n = 1, \cdots, r$, and scalars $\delta_i > 0$, $i = 1, 2$ such that*

$$\begin{bmatrix} \Upsilon^p_{ijklmn} & \Gamma_{ijl} \\ * & \bar{\Omega}_{jkl} \end{bmatrix} < 0, \quad i, j, k, l, m, n = 1, \cdots, r, \; p = 1, \cdots, 4, \tag{4.3}$$

$$\delta_1 - \delta_2 > 0 \tag{4.4}$$

$$\begin{bmatrix} \frac{1}{\tau_1}\bar{Z}_i + \bar{X}_{2mn} & -\bar{X}_{2mn} \\ -\bar{X}_{2mn} & \bar{Q}_{1imn} + \bar{X}_{2mn} \end{bmatrix} \geq 0 \; i, m, n = 1, \cdots, r, \tag{4.5}$$

$$\begin{bmatrix} \frac{1}{\tau_2-\tau_1}\bar{Z}_i + \bar{X}_{4mn} & -\bar{X}_{4mn} \\ -\bar{X}_{4mn} & \bar{Q}_{2imn} + \bar{X}_{4mn} \end{bmatrix} \geq 0 \; i, m, n = 1, \cdots, r \tag{4.6}$$

*where*

$$\begin{aligned}
\Upsilon^1_{ijklmn} &= \tfrac{1}{2}\bar{\theta}_{ijklmn} + \bar{\theta}_{1jmn} + \delta_1 I, \\
\Upsilon^2_{ijklmn} &= \tfrac{1}{2}\bar{\theta}_{ijklmn} + \bar{\theta}_{2jmn} + \delta_1 I, \\
\Upsilon^3_{ijklmn} &= \tfrac{1}{2}\bar{\theta}_{ijklmn} + \bar{\theta}_{3jmn} - \delta_2 I, \\
\Upsilon^4_{ijklmn} &= \tfrac{1}{2}\bar{\theta}_{ijklmn} + \bar{\theta}_{4jmn} - \delta_2 I, \\
\bar{\theta}_{1jmn} &= -e_7^T\bar{X}_{3jmn}e_7 - (e_2-e_4)^T\bar{X}_{4mn}(e_2-e_4), \\
\bar{\theta}_{2jmn} &= -e_6^T\bar{X}_{3jmn}e_6 - (e_2-e_3)^T\bar{X}_{4mn}(e_2-e_3), \\
\bar{\theta}_{3jmn} &= -e_{10}^T\bar{R}_{3jmn}e_{10} - (\tau_{12}e_1-e_7)^T\bar{R}_{4mn}(\tau_{12}e_1-e_7), \\
\bar{\theta}_{4jmn} &= -e_9^T\bar{R}_{3jmn}e_9 - (\tau_{12}e_1-e_6)^T\bar{R}_{4mn}(\tau_{12}e_1-e_6), \\
\bar{\theta}_{ijlkmn} &= \bar{\pi}_{ijlkmn} - e_5^T\bar{X}_{1mn}e_5 - (e_1-e_3)^T\bar{X}_{2mn}(e_1-e_3) - e_6^T\bar{X}_{3jmn}e_6 - e_7^T\bar{X}_{3jmn}e_7 - (e_2-e_3)^T\bar{X}_{4mn}(e_2-e_3) \\
&\quad -(e_2-e_4)^T\bar{X}_{4mn}(e_2-e_4) - e_8^T\bar{R}_{1jmn}e_8 - (\tau_1e_1-e_5)^T\bar{R}_{2mn}(\tau_1e_1-e_5) - e_9^T\bar{R}_{3jmn}e_9 \\
&\quad -e_{10}^T\bar{R}_{3jmn}e_{10} - (\tau_{12}e_1-e_6)^T\bar{R}_{4mn}(\tau_{12}e_1-e_6) - (\tau_{12}e_1-e_7)^T\bar{R}_{4mn}(\tau_{12}e_1-e_7) \\
&\quad -(\tfrac{\tau_{12}}{2}e_1-e_8)^T\bar{U}_{1mn}(\tfrac{\tau_{12}}{2}e_1-e_8) - (\tfrac{\tau_{12}^{(2)}}{2}e_1-e_9-e_{10})^T\bar{U}_{2mn}(\tfrac{\tau_{12}^{(2)}}{2}e_1-e_9-e_{10})
\end{aligned}$$

$$\bar{\Xi}_{kl} = -\bar{X}_{2kl} - \tau_1^2\bar{R}_{2kl} - 2\tau_{12}^2\bar{R}_{4kl} - \frac{\tau_1^4}{4}\bar{U}_{1kl} - \frac{(\tau_{12}^{(2)})^2}{4}\bar{U}_{2kl},$$

$$\bar{\pi}_{ijklmn} = \begin{bmatrix}
\bar{\Lambda}_{ijkl} & B_iK_l & 0 & 0 & P_{1mn} & 0 & \tau_1\bar{P}_{3mn} & \tau_{12}\bar{P}_{4mn} & \tau_{12}\bar{P}_{4mn} & 0 & D_i \\
* & -(1-\eta)W_{jmn} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & -\bar{Q}_{1jmn}+\bar{Q}_{2jmn} & 0 & -P_{1mn} & \bar{P}_{2mn} & \bar{P}_{2mn} & 0 & 0 & 0 & 0 \\
* & * & * & -\bar{Q}_{2jmn} & 0 & -\bar{P}_{2mn} & -\bar{P}_{2mn} & 0 & 0 & 0 & 0 \\
* & * & * & * & 0 & 0 & 0 & -\bar{P}_{3mn} & 0 & 0 & 0 \\
* & * & * & * & * & 0 & 0 & 0 & -\bar{P}_{4mn} & -\bar{P}_{4mn} & 0 \\
* & * & * & * & * & * & 0 & 0 & -\bar{P}_{4mn} & -\bar{P}_{4mn} & 0 \\
* & * & * & * & * & * & * & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -\gamma^2 I
\end{bmatrix},$$

$$
\begin{aligned}
\bar{\Lambda}_{ijkl} &= A_iZ_j + Z_jA_i^T + \bar{Q}_{1jkl} + \bar{W}_{jkl} - \tfrac{1}{h}(\bar{Z}_i - \bar{Z}_l) + \tau_1^2\bar{X}_{1jkl} + \tau_{12}^2\bar{X}_{3jkl} + \tfrac{\tau_1^4}{4}\bar{R}_{1jkl} + \tfrac{(\tau_{12}^{(2)})^2}{4}\bar{R}_{3jkl},\\
\Gamma_{ijl}^T &= \begin{bmatrix} A_iZ_j & B_iK_l & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_iZ_j\\ C_iZ_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\\
\bar{\Omega}_{jkl} &= \begin{bmatrix} 2(-2Z_j + \tau_1^2\bar{X}_{2kl} + \tau_{12}^2\bar{X}_{4kl} + \tfrac{\tau_1^4}{4}\bar{R}_{2kl} + \tfrac{(\tau_{12}^{(2)})^2}{4}\bar{R}_{4kl} + \tfrac{\tau_1^6}{36}\bar{U}_{1kl} + \tfrac{(\tau_2^3-\tau_1^3)^2}{36}\bar{U}_{2kl}) & 0\\ * & -I \end{bmatrix}
\end{aligned}
$$

*where $\tau_{12} = \tau_2 - \tau_1,\ \tau_{12}^{(2)} = \tau_2^2 - \tau_1^2$. In this case, control gains $K_l$ and $Z_j,\ j,l = 1,\cdots,r$ can be found as solutions of the above LMIs.*

**Proof:** We consider the same Lyapunov-Krasovskii function (3.9) except for the first term of $V_1(x_t)$, which is replaced by

$$\bar{V}_1(x_t) = x^T(t)Z_\mu^{-1}x(t).$$

The time-derivative of $V_{11}(x_t)$ is calculated as

$$
\begin{aligned}
\dot{\bar{V}}_1(x_t) &= 2x^T(t)Z_\mu^{-1}(A_\lambda x(t) + B_\lambda K_\mu^\tau(Z_\mu^\tau)^{-1}x(t-\tau(t)) + D_\lambda w(t)) + x^T(t)\dot{Z}_\mu^{-1}x(t)\\
&= x^T(t)Z_\mu^{-1}(A_\lambda Z_\mu + Z_\mu A_\lambda^T - \dot{Z}_\mu)Z_\mu^{-1}x(t) + 2x^T(t)Z_\mu^{-1}B_\lambda K_\mu^\tau(Z_\mu^\tau)^{-1}x(t-\tau(t)))\\
&\quad +2x^T(t)Z_\mu^{-1}D_\lambda w(t))
\end{aligned}
$$

We follow the similar lines of proof of Theorem 3.1, and obtain

$$
\begin{aligned}
\dot{V}(x_t) = &\sum_{i=1}^r\sum_{j=1}^r\sum_{l=1}^r \lambda_i(\xi)\mu_j(\xi)\mu_k(\xi)\mu_l(\xi)\mu_m(\xi(t-\tau))\\
&\times\mu_n(\xi(t-\tau))\bar{\zeta}^T(t)(\alpha\bar{\theta}_{ijklmn}^{(1)} + (1-\alpha)\bar{\theta}_{ijklmn}^{(2)} + \varepsilon\bar{\theta}_{ijklmn}^{(3)} + (1-\varepsilon)\bar{\theta}_{ijklmn}^{(4)})\bar{\zeta}(t)
\end{aligned}
$$

where $\bar{\theta}_{ijklmn}^{(p)} = \frac{1}{2}\tilde{\theta}_{ijklmn} + \bar{\theta}_{pjmn},\ p = 1,2,\ \bar{\theta}_{ijklmn}^{(p)} = \frac{1}{2}\tilde{\theta}_{ijklmn} + \bar{\theta}_{pjmn},\ p = 3,4,$

$$
\tilde{\theta}_{ijklmn} = \bar{\theta}_{ijklmn} + \Phi_{il}^T\Omega\Phi_{il} + \begin{bmatrix} Z_\mu^TC_\lambda^TC_\lambda Z_\mu & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 \end{bmatrix},
$$

$$\bar{\zeta} = \begin{bmatrix} Z_\mu^{-1} & (Z_\mu^\tau)^{-1} & \cdots & (Z_\mu^\tau)^{-1} & I \end{bmatrix}\zeta.$$

We have defined the following matrices:

$$
\begin{aligned}
&\sum_{j=1}^r\sum_{k=1}^r\sum_{l=1}^r \mu_j(\xi)\mu_k(\xi)\mu_l(\xi)\bar{Q}_{jkl} = Z_\mu Q_\mu Z_\mu,\\
&\sum_{j=1}^r\sum_{k=1}^r\sum_{l=1}^r \mu_j(\xi)\mu_m(\xi(t-\tau(t)))\mu_n(\xi(t-\tau(t)))\bar{Q}_{jmn} = Z_\mu^\tau Q_\mu Z_\mu^\tau
\end{aligned}
$$

for example. Similar notations have also been used for others matrices. Applying the Schur complement formula and the inequality $-\Omega^{-1} \le -2Z + Z\Omega Z$, we obtain (4.3)-(4.6).

**Remark 4.1.** *In case that the delay rate $\eta$ is unknown, we can still make use of Theorem 4.1 with $W_j = 0,\ j = 1,\cdots,r$.*

**Remark 4.2.** *The conditions (4.3)-(4.6) are not strict LMIs unless $h > 0$ is given, either. However, they can be solved in the same way as discussed in Remark 3.3.*

## 5. Numerical Example

We consider the system([19])

$$\dot{x}(t) = \sum_{i=1}^{2} \lambda_i(\xi)\{A_i x(t) + B_i u(t) + D_i w(t)\}, \tag{5.1}$$

$$z(t) = \sum_{i=1}^{2} \lambda_i(\xi) C_i x(t) \tag{5.2}$$

where $x_1(t) \in [1,\ -1]$ and

$$A_1 = \begin{bmatrix} 0 & 1 \\ -0.01 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 1 \\ -0.68 & 0 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0.1 \end{bmatrix},\ C_2 = \begin{bmatrix} 1.1 & 0.1 \end{bmatrix},\ D_1 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix},$$

$$\lambda_1(x_1) = 1 - x_1^2, \lambda_2(x_1) = x_1^2.$$

Suppose that $0.0 \le \tau(t) \le 1.50$ and $\eta = 0.3$.

First, we compare our results with others to show the effectiveness of Theorem 3.1 for stabilization with $w(t) = 0$.

| Method | $\tau_2$ |
|---|---|
| [19] | 0.60 |
| [7] | 1.40 |
| Theorem 3.1 | 2.38 |

Table 1: Comparison of the methods

This obviously show that our new multiple Lyapunov-Krasovskii function method is better than the existing conditions.

Next, we design an $H_\infty$ controller for the fuzzy networked system (5.1)-(5.2). Given the $H_\infty$ attenuation level $\gamma = 1$, Theorem 4.1 gives the feedback control $u(t)$ by

$$u(t) = K_\mu^\tau (Z_\mu^\tau)^{-1} x(t - \tau(t)) \tag{5.3}$$

where

$$K_1 = \begin{bmatrix} 0.0522 & -0.1368 \end{bmatrix},\ K_2 = \begin{bmatrix} 0.1121 & -0.1643 \end{bmatrix},$$
$$Z_1 = \begin{bmatrix} 0.0936 & -0.0485 \\ -0.0485 & 0.1289 \end{bmatrix},\ Z_2 = \begin{bmatrix} 0.0924 & -0.0468 \\ -0.0468 & 0.1258 \end{bmatrix}.$$

Theorem 4.1 is based on Theorem 3.1, which has been shown to be least conservative in the above numerical example. It implies that Theorem 4.1 is a control design method which requires less conservative design conditions than others.

Finally, the simulation result on the state trajectories of the closed-loop system with the initial conditions $x(0) = [-0.5\ \ 0.5]^T$ and the zero-mean Gaussian random variable $w(t)$ of variance 0.1 is shown in Figure 1. The delay $\tau(t)$ is assumed to be $\tau(t) = 1 + 0.5\sin(0.1t)$. The bold and dotted lines indicate $x_1(t)$ and $x_2(t)$, respectively, and they show the system stability with disturbance attenuation.

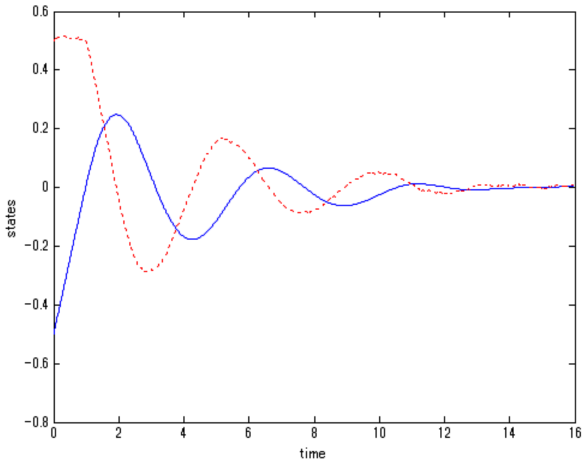


**Figure 1.** The state trajectories

## 6. Conclusions

The $H_\infty$ disturbance attenuation and control design of nonlinear networked control systems described by Takagi-Sugeno fuzzy systems have been considered. A new multiple Lyapunov-Krasovskii function was introduced to obtain new $H_\infty$ disturbance attenuation conditions for the closed-loop system. This technique leads to less conservative conditions. Control design method for nonlinear networked control systems was also proposed based on the same multiple Lyapnov-Krasovskii function and thus conditions for control design are less conservative than the existing ones.

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